Control systems and locally convex topologies[1]

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Introduction

- In geometric control theory, a control system is described by the following differential equation

$$\dot{x} = f(u, x),$$

where the right hand side is a parametrized family of vector fields $f : \mathcal{U} \times M \to TM$, with $\mathcal{U}$ being the control set.

- The trajectories of the control system are the solutions of this differential equation for a locally essentially bounded control $u(\cdot)$.

Introduction

- In the literature, there are many different regularity assumptions on f.
- In one approach,[3] it is assumed that the control set $\mathcal{U}$ is a topological space and the parametrized vector field $f : \mathcal{U} \times M \to TM$ has first derivatives continuous with respect to x and u.
- Although this is a general and coherent approach, but it has the deficiency of not accounting for stronger regularity when it is present.

[3] For example in the book "Mathematical Control Theory" by Sontag

- In another approach,[4] it is assumed that the control set $\mathcal{U}$ is an open subset of Euclidean space and the parametrized vector field $f : \mathcal{U} \times M \to TM$ is of class C^ν, for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$.
- This approach includes general regularity classes, but it is restrictive in terms of control sets (the control set is an open subset of $\mathbb{R}^k$).
- It seems that there is no coherent approach for studying different regularity classes of control systems in the literature.

[4] For example in the book "Foundation of Optimal Control Theory" by E. B. Lee & L. Markus

Introduction

- In this talk we give a unified framework for studying regularity class C^ν, for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$.
- In particular, our framework includes the real analytic class.
- In order to construct such a framework, we first assume that
  1. The control set $\mathcal{U}$ is an arbitrary topological space.
  2. The parametrized vector field $f : \mathcal{U} \times M \to TM$ is of class C^ν with respect to x, when u is fixed.
- We call this a C^ν-**parametrized vector field**.

Space of vector fields

- **Idea**: Consider C^{ν}-parametrized vector fields f as maps from the space of parameters to the space of vector fields.
- We denote by $\Gamma^{\nu}(TM)$ the set of all vector fields of class C^{ν} on M.
- $\Gamma^{\nu}(TM)$ is a **vector space**.

Correspondence

If $f : \mathcal{U} \times M \to TM$ is a C^{ν}-parametrized vector field, then the corresponding map $\widehat{f} : \mathcal{U} \to \Gamma^{\nu}(TM)$ is defined as

$$\widehat{f}(u)(x) = f(u, x).$$

- In order to impose useful conditions on $\widehat{f}$, we will use a topology on $\Gamma^{\nu}(TM)$.

Locally convex topologies

Locally convex space

A **locally convex space**, is a vector space V equipped with a family of seminorms $\{p_\alpha\}_{\alpha \in A}$.

- **Comparison**: Locally convex spaces can be considered as a generalization of normed spaces.
- Similar to normed spaces, one can define a topology on a locally convex space using seminorms.
- One can define similar notions such as boundedness, continuity and measurability for locally convex spaces.

Normed topology

Suppose that V is a vector space with norm $\|\cdot\|$ and U is a topological space.

Continuity

A map $f : U \to V$ is **continuous at** $u \in U$ if for every $\epsilon > 0$, there exists a neighbourhood N_u of u such that

$$\|f(v) - f(u)\| < \epsilon, \qquad \forall v \in N_u$$

Locally convex topology

Suppose that V is a locally convex space with seminorms $\{p_\alpha\}_{\alpha \in A}$ and U is a topological space.

Continuity

A map $f : U \to V$ is **continuous at** $u \in U$ if for every $\alpha \in A$ and every $\epsilon > 0$, there exists a neighbourhood N_u of u such that

$$p_\alpha(f(v) - f(u)) < \epsilon, \qquad \forall v \in N_u$$

Locally convex topologies on space of vector fields

We define a locally convex structure on $\Gamma^\nu(TM)$ using a family of seminorms.

- For defining the locally convex structure on $\Gamma^\nu(TM)$, we separate the cases $\nu \in \mathbb{Z}_{>0}$, $\nu = \infty$ and $\nu = \omega$.
- If $\xi \in \Gamma^\nu(TM)$, then $j_m\xi(x)$ can be considered as the first m terms in Taylor series of ξ around x.
- We define a fiber norm $\|\cdot\|$ on the space of jets in a specific way (not presented here).
- We define $\mathbf{c_0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ as

$$\mathbf{c_0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}) = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}_{\geq 0}, \ \lim_{i\to\infty} a_i = 0\}.$$

Locally convex topologies on space of vector fields

The CO^ν-structure on $\Gamma^\nu(TM)$ is the locally convex structure on $\Gamma^\nu(TM)$ defined using the seminorms,

Cases $\nu \in \mathbb{Z}_{>0}$

$$p_K^\nu(\xi) = \sup\{\|j_\nu \xi(x)\| \mid x \in K\}, \ K \subseteq M \text{ compact};$$

Case $\nu = \infty$

$$p_{K,m}^\infty(\xi) = \sup\{\|j_m \xi(x)\| \mid x \in K\}, \ m \in \mathbb{Z}_{\geq 0}, \ K \subseteq M \text{ compact};$$

Case $\nu = \omega$

$$\begin{aligned} p_{K,\mathbf{a}}^\omega(\xi) = \sup\{a_0 a_1 \dots a_m \|j_m \xi(x)\| \mid x \in K, m \in \mathbb{Z}_{\geq 0}\}, \\ \mathbf{a} = (a_0, a_1, \dots) \in \mathbf{c_0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}), \ K \subseteq M \text{ compact.} \end{aligned}$$

C^ν-control systems

- Using the CO^ν-topology, we can define a C^ν-control system as

C^ν-control system

A C^ν-control system is a triple $\Sigma = (M, f, \mathcal{U})$, where

1. M is a differentiable manifold,
2. $\mathcal{U}$ is a topological space, and
3. $f : \mathcal{U} \times M \to TM$ is a C^ν-parametrized vector field such that $\widehat{f} : \mathcal{U} \to \Gamma^\nu(TM)$ is continuous in CO^ν-topology.

- The third condition is a checkable condition, using the seminorms for CO^ν-topology on $\Gamma^\nu(TM)$.

Main Theorem

These CO^{ν}-topologies helps us to prove the following fundamental result.

Theorem

Consider the control system

$$\dot{x} = f(u, x),$$

where $f : \mathcal{U} \times M \to TM$ is a C^{ν}-parametrized vector field for $\nu \in \mathbb{Z}_{>0} \cup \{\infty\} \cup \{\omega\}$. If the curve $\hat{f} : \mathcal{U} \to \Gamma^{\nu}(TM)$ is **continuous** in CO^{ν}-topology on $\Gamma^{\nu}(TM)$, then the trajectory of the system starting at x_0 **exists**, is **unique** and is $C^{\nu-1}$ **dependent on the** x_0.

This result relies on a deep and difficult theorem about time-varying vector fields[5].

[5] S. Jafarpour and A. D. Lewis. "Mathematical models for geometric control theory". In: *ArXiv e-prints* (Dec. 2013). arXiv:1312.6473 [math.OC].

A classical result

One can show that for $\nu = 1$, our main theorem is just the classical existence and uniqueness result for $M = \mathbb{R}^n$.

Existence and Uniqueness Theorem

Suppose that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-parametrized vector field on $\mathbb{R}^n$, $\mu : \mathbb{T} \to \mathbb{R}$ is a locally essentially bounded curve and $M, N_1, N_2, \ldots, N_n > 0$ such that

$$\begin{aligned} |f(\mu(t), \mathbf{x})| &\leq M, \\ \left| \frac{\partial f}{\partial x^j}(\mu(t), \mathbf{x}) \right| &\leq N_j, \quad \forall j \in \{1, 2, \ldots, n\}, \end{aligned}$$

holds for almost every t, in a neighbourhood of x_0. Then the trajectory of the system for the control μ starting at x_0 exists, is unique and depends continuously on the initial condition.

Example

Control-affine systems with vector fields in $\Gamma^\nu(TM)$ are C^ν-control systems.

Example

Consider a control-affine system with $f : \mathbb{R}^m \times M \to TM$ defined as

$$f(u, x) = f_0(x) + \sum_{i=1}^{m} u^i f_i(x),$$

such that $f_i \in \Gamma^\nu(TM)$ for every $i \in \{0, 1, \ldots, m\}$.
One can show that $\widehat{f} : \mathbb{R}^m \to \Gamma^\nu(TM)$ is continuous in CO^ν-topology, so $\Sigma = (M, f, \mathbb{R}^m)$ is a C^ν-control system. So trajectories for control-affine systems depend in a regular manner on initial conditions when an open-loop control has been fixed.