Reachability Analysis of Dynamical Systems:

A Mixed Monotone Contracting Approach

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UCSB

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UCSB

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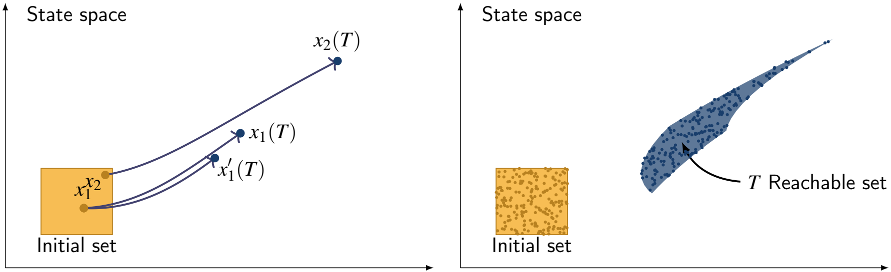
SJ and S. Coogan. Monotonicity and contraction on polyhedral cones. arXiv, 2023.

A. Davydov and **SJ** and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. IEEE TAC, 2022

Outline of this talk

- **Reachability Analysis**
- Contraction-based Reachability
- Mixed Monotone Reachability

System : $\dot{x} = f(x, w)$ **State** : $x \in \mathbb{R}^n$ **Uncertainty** : $w \in \mathcal{W} \subseteq \mathbb{R}^m$



What are the possible states of the system at time T?

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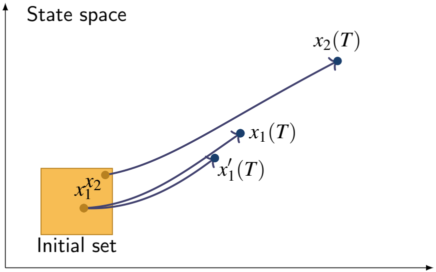

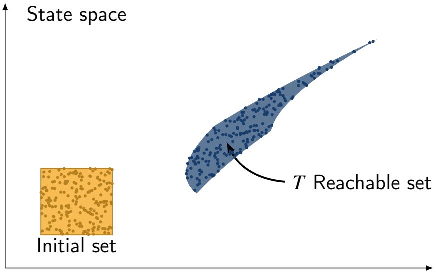

What are the possible states of the system at time T?

- **T-reachable sets** characterize evolution of the system

$$\mathcal{R}_f(T, \mathcal{X}_0, \mathcal{W}) = \{x_w(T) \mid x_w(\cdot) \text{ is a traj for some } w(\cdot) \in \mathcal{W} \text{ with } x_0 \in \mathcal{X}_0\}$$

A large number of **safety specifications** can be represented using T-reachable sets

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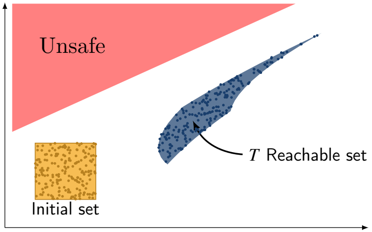
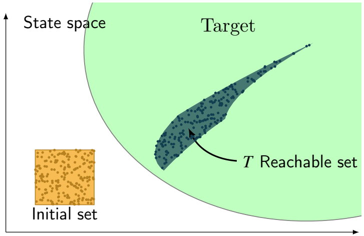
- Example: Reach-avoid problem





$$\mathcal{R}_f(T, \mathcal{X}_0, \mathcal{W}) \cap \text{ Unsafe set } = \emptyset$$

$$\mathcal{R}_f(T_{\text{final}}, \mathcal{X}_0, \mathcal{W}) \subseteq \text{Target set}$$

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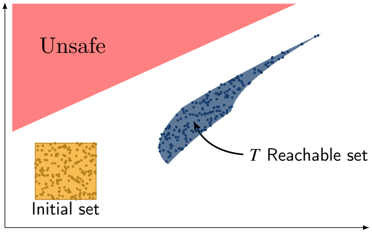

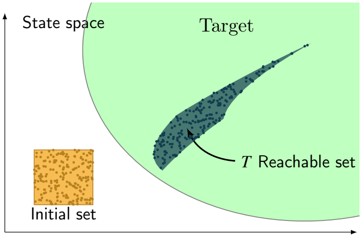

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Combining different instantiation of Reach-avoid problem $\Longrightarrow$
diverse range of specifications
(complex planning using logics, invariance, stability)

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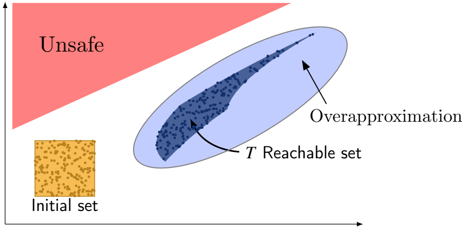

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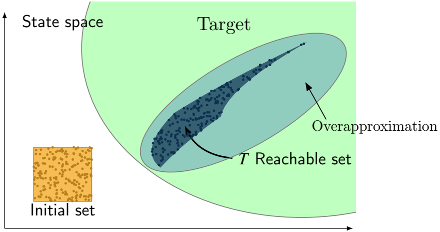

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Different approaches for approximating reachable sets

- Linear, and piecewise linear systems (Ellipsoidal methods) (Kurzhanski and Varaiya, 2000)
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In this talk: use control theoretic tools to develop computationally efficient approaches for reachability

Outline of this talk

- Reachability Analysis
- **Contraction-based Reachability**
- Mixed Monotone Reachability

Contraction Theory

From stability to reachability

$\dot{x} = f(x, w)$ is contracting wrt $\|\cdot\|$ with rate c if
the dist between every two traj is decreasing/increasing with exp rate c wrt $\|\cdot\|$

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Matrix measure

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a norm $\|\cdot\|$:

$$\mu_{\|\cdot\|}(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

Given $\eta \in \mathbb{R}^n_{\geq 0}$

$$\mu_{2,\eta}(A) = \tfrac{1}{2}\lambda_{\mathsf{max}}(\mathrm{diag}(\eta)A + A^\top \mathrm{diag}(\eta))$$

$$\mu_{1,\eta}(A) = \max_j \left(a_{jj} + \sum\nolimits_{i\neq j} |a_{ij}|\tfrac{\eta_j}{\eta_i}\right)$$

$$\mu_{\infty,\eta}(A) = \max_i \left(a_{ii} + \sum\nolimits_{j\neq i} |a_{ij}|\tfrac{\eta_j}{\eta_i}\right)$$

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- directional derivative of matrix norm $\|\cdot\|$ in direction of A at point I_n,
- **In the literature**: one-sided Lipschitz constant, logarithmic norm

Assume $\mu_{\|\cdot\|}\left(\frac{\partial f}{\partial x}(x,w)\right) \leq c$ and $\left\|\frac{\partial f}{\partial w}(x,w)\right\| \leq \ell$

ISS bound: $\|x(t) - x^*(t)\| \leq e^{ct}\|x(0) - x^*(0)\| + \frac{\ell(e^{ct}-1)}{c}\|w(t) - w^*\|$

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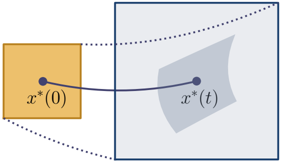
Theorem (classical)

If $\mathcal{X}_0 = B_{\|\cdot\|}(r_1, x^*(0))$ and $\mathcal{W} = B_{\|\cdot\|}(r_2, w^*)$, then

$$\mathcal{R}_f(t, \mathcal{X}_0) \subseteq B_{\|\cdot\|}(e^{ct}r_1 + \tfrac{\ell}{c}(e^{ct}-1)r_2, x^*(t))$$

where $x^*(\cdot)$ is the solution of $\dot{x} = f(x, w^*)$ with $x(0) = x^*(0)$.



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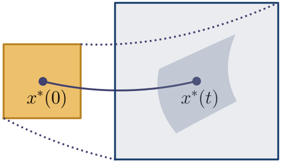
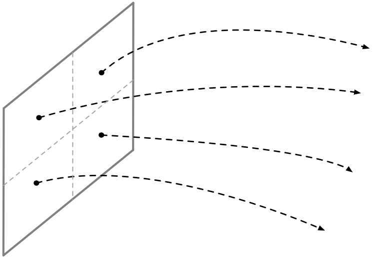


- proof is based on generalized version of Grönwall's lemma
- sharper results using time-varying and locally-defined c and ℓ

Simulation-based Reachability

Contraction tubes

- **system's simulations** to improve accuracy of reachability
- **contraction theory** to provide guarantees for reachability



- cover the initial set $\mathcal{X}_0$ and the disturbance set $\mathcal{W}$ with $\|\cdot\|$-norm balls[1]
- pick a sample point in each covering

[1] Fan et. al., *Simulation-Driven Reachability Using Matrix Measures*, 2017

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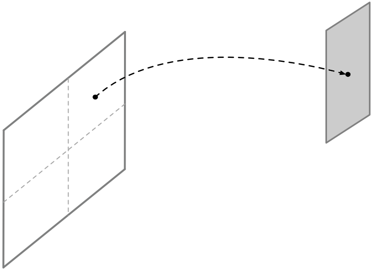

- compute reach tube $B_{\|\cdot\|}(e^{ct} r_1 + \frac{\ell}{c}(e^{ct} - 1) r_2, x^*(t))$
- all trajectories starting in the covering remain in the reach tube

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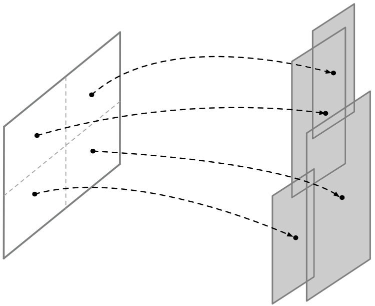

over-approximation of the reachable set = union of $\|\cdot\|$-norm balls

$$B_{\|\cdot\|}(e^{ct}r_1 + \tfrac{\ell}{c}(e^{ct}-1)r_2, x^*(t))$$

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Outline of this talk

- Reachability Analysis
- Contraction-based Reachability
- **Mixed Monotone Reachability**

A dynamical system $\dot{x} = f(x, w)$ is monotone[2] if

$$x_u(0) \leq y_w(0) \quad \text{and} \quad u \leq w \quad \Longrightarrow \quad x_u(t) \leq y_w(t) \quad \text{for all time}$$

where $\leq$ is the component-wise partial order.

[2] Angeli and Sontag, "Monotone control systems", IEEE TAC, 2003

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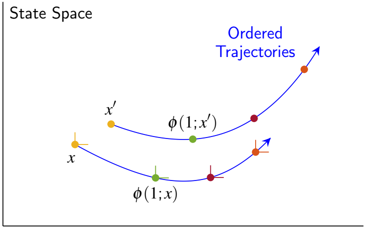
Monotonicity test

1. $\frac{\partial f}{\partial x}(x,w)$ is Metzler (off-diag ≥ 0)
2. $\frac{\partial f}{\partial w}(x,w) \geq 0$



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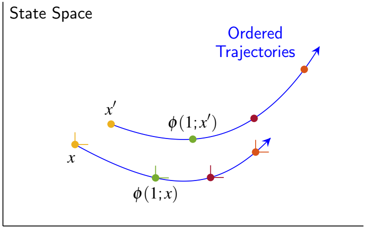

In this talk: monotone system theory for reachability analysis

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Theorem (classical result)

For a monotone system with $\mathcal{W} = [\underline{w}, \overline{w}]$

$$\mathcal{R}_f(t, [\underline{x}_0, \overline{x}_0]) \subseteq [x_{\underline{w}}(t), x_{\overline{w}}(t)]$$

where $x_{\underline{w}}(\cdot)$ (resp. $x_{\overline{w}}(\cdot)$) is the trajectory with disturbance $\underline{w}$ (resp. $\overline{w}$) starting at $\underline{x}_0$ (resp. $\overline{x}_0$)

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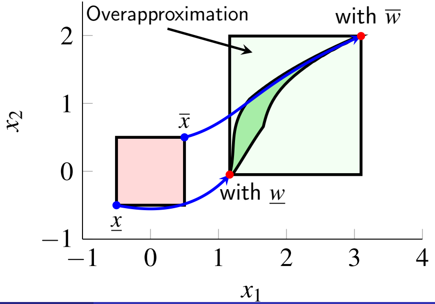
Example:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_1 + w \\ x_1 \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$



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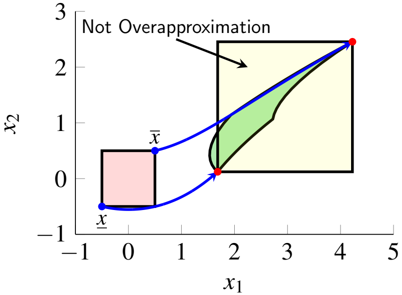

- **Key idea:** embed the dynamical system on $\mathbb{R}^n$ into a dynamical system on $\mathbb{R}^{2n}$
- Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$

Original system

$$\dot{x} = f(x, w)$$

Embedding system

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}),$$
$$\dot{\overline{x}} = \overline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$$

$\underline{d}, \overline{d}$ are **decomposition functions** s.t.

1. $f(x, w) = \underline{d}(x, x, w, w)$ for every x, w
2. cooperative: $(\underline{x}, \underline{w}) \mapsto \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$
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The embedding system is a monotone dynamical system on $\mathbb{R}^{2n}$ with respect to the **southeast** partial order $\leq_{\mathrm{SE}}$:

$$\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} y \\ \widehat{y} \end{bmatrix} \quad \Longleftrightarrow \quad x \leq y \quad \text{and} \quad \widehat{y} \leq \widehat{x}$$

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J-L. Gouze and L. P. Hadeler. Monotone flows and order intervals. Nonlinear World, 1994

G. Enciso, H. Smith, and E. Sontag. Nonmonotone systems decomposable into monotone systems with negative feedback . Journal of Differential Equations, 2006.

H. Smith. Global stability for mixed monotone systems. Journal of Difference Equations and Applications, 2008

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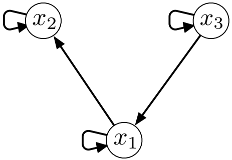
In this talk: mixed monotone theory for reachability analysis

- **Metzler/non-Metzler** decomposition: $A = \lceil A \rceil^{\mathrm{Mzl}} + \lfloor A \rfloor^{\mathrm{Mzl}}$
- Example: $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \lceil A \rceil^{\mathrm{Mzl}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \lfloor A \rfloor^{\mathrm{Mzl}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Linear systems

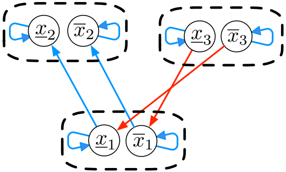
Original system

$$\dot{x} = Ax + Bw$$



Embedding system

$$\dot{\underline{x}} = \lceil A \rceil^{\mathrm{Mzl}} \underline{x} + \lfloor A \rfloor^{\mathrm{Mzl}} \overline{x} + B^{+} \underline{w} + B^{-} \overline{w}$$
$$\dot{\overline{x}} = \lceil A \rceil^{\mathrm{Mzl}} \overline{x} + \lfloor A \rfloor^{\mathrm{Mzl}} \underline{x} + B^{+} \overline{w} + B^{-} \underline{w}$$

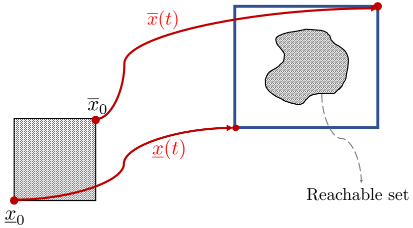


Theorem[3]

Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ and

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}), \qquad \underline{x}(0) = \underline{x}_0$$
$$\dot{\overline{x}} = \overline{d}(\overline{x}, \underline{x}, \overline{w}, \underline{w}), \qquad \overline{x}(0) = \overline{x}_0$$

Then $\mathcal{R}_f(t, \mathcal{X}_0) \subseteq [\underline{x}(t), \overline{x}(t)]$



[3] Coogan and Arcak, "Efficient finite abstraction of mixed monotone systems", HSCC, 2015.

Theorem[3]

Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ and

$$\begin{aligned} \dot{\underline{x}} &= \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}), & \underline{x}(0) &= \underline{x}_0 \\ \dot{\overline{x}} &= \overline{d}(\overline{x}, \underline{x}, \overline{w}, \underline{w}), & \overline{x}(0) &= \overline{x}_0 \end{aligned}$$

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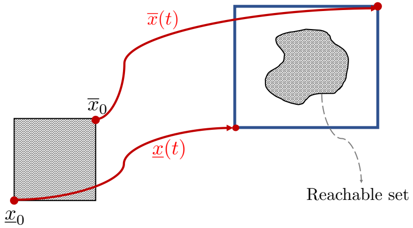

(Scalable) a single trajectory of embedding system provides **lower bound** $(\underline{x})$ and **upper bound** $(\overline{x})$ for the trajectories of the original system.

[3] Coogan and Arcak, "Efficient finite abstraction of mixed monotone systems", HSCC, 2015.

Original System:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_2 + w \\ x_1 \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$

blue = cooperative, red = competitive

Decomposition function

$$\underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \begin{bmatrix} \underline{x}_2^3 + \underline{w} \\ \underline{x}_1 \end{bmatrix} + \begin{bmatrix} -\overline{x}_2 \\ 0 \end{bmatrix}$$

$$\overline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \begin{bmatrix} \overline{x}_2^3 + \overline{w} \\ \overline{x}_1 \end{bmatrix} + \begin{bmatrix} -\underline{x}_2 \\ 0 \end{bmatrix}$$

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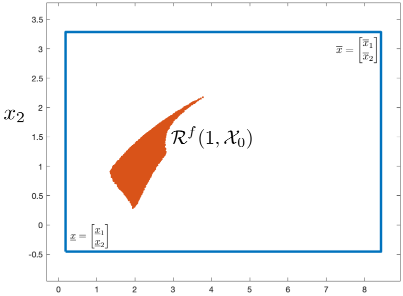
Embedding System:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = \begin{bmatrix} \underline{x}_2^3 - \overline{x}_2 + \underline{w} \\ \underline{x}_1 \\ \overline{x}_2^3 - \underline{x}_2 + \overline{w} \\ \overline{x}_1 \end{bmatrix} \qquad \begin{bmatrix} \underline{w} \\ \overline{w} \end{bmatrix} = \begin{bmatrix} 2.2 \\ 2.3 \end{bmatrix}$$

$$\begin{bmatrix} \underline{x}_1(0) \\ \underline{x}_2(0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \qquad \begin{bmatrix} \overline{x}_1(0) \\ \overline{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$



How to compute a decomposition function for a system?

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Different approaches for constructing decomposition functions

- linear systems
- polynomial systems
- bounded Jacobian

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Every **locally Lipschitz** system has at least one decomposition function

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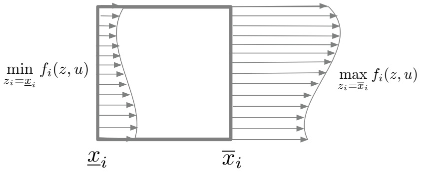
Every **locally Lipschitz** system has at least one decomposition function

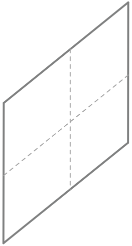
The best (tightest) decomposition function is given by

$$\underline{d}_i(\underline{x},\overline{x},\underline{w},\overline{w}) = \min_{\substack{z\in[\underline{x},\overline{x}],z_i=\underline{x}_i\\ u\in[\underline{w},\overline{w}]}} f_i(z,u),$$

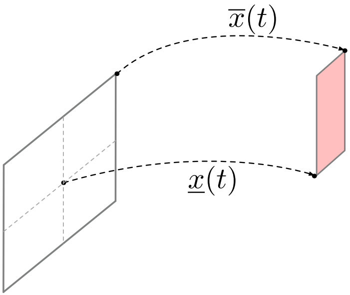
$$\overline{d}_i(\underline{x},\overline{x},\underline{w},\overline{w}) = \max_{\substack{z\in[\underline{x},\overline{x}],z_i=\overline{x}_i\\ u\in[\underline{w},\overline{w}]}} f_i(z,u)$$



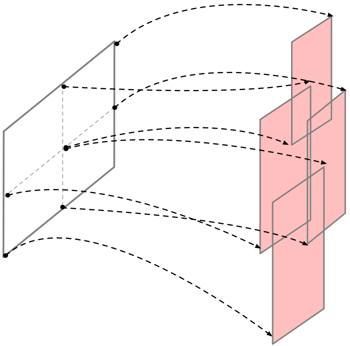
cover the initial set $\mathcal{X}_0$ and the disturbance set $\mathcal{W}$ with hyper-rectangles



For each covering, simulate a single trajectory of the embedding system



Union of hyper-rectangles = over-approximation of the reachable set



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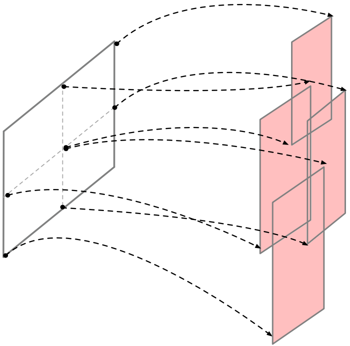
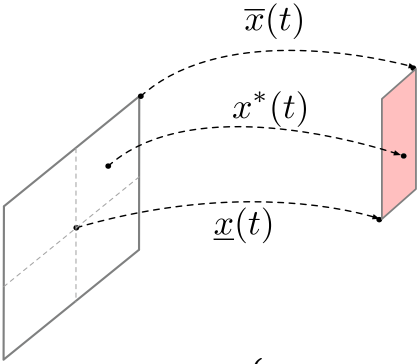


Question: how accurate is mixed monotone reachability?



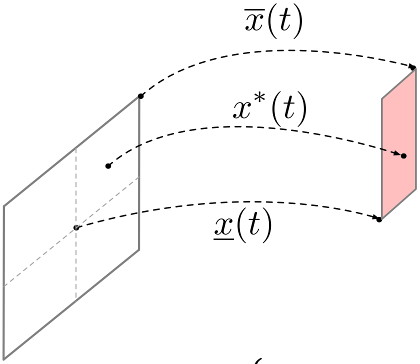
$$\begin{bmatrix} x^*(t) \\ x^*(t) \end{bmatrix} \text{ and } \begin{bmatrix} \underline{x}(t) \\ \overline{x}(t) \end{bmatrix} \text{ traj of embedding system } \implies \begin{cases} \|x^*(t) - \underline{x}(t)\|_\infty \le e^{dt}\|x^*(0) - \underline{x}(0)\|_\infty \\ \|x^*(t) - \overline{x}(t)\|_\infty \le e^{dt}\|x^*(0) - \overline{x}(0)\|_\infty \end{cases}$$

Question: how accurate is mixed monotone reachability?

Accuracy $=$ the incremental distance between trajectories of **embedding system**

Simulation-based Reachability

A mixed monotone approach



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Question: what is the contraction rate of the embedding system?

Theorem[4]

Let $\frac{d}{dt}\begin{bmatrix}\underline{x}\\ \overline{x}\end{bmatrix} = \begin{bmatrix}\underline{d}(\underline{x},\overline{x},\underline{w},\overline{w})\\ \overline{d}(\underline{x},\overline{x},\underline{w},\overline{w})\end{bmatrix} := e(\underline{x},\overline{x},\underline{w},\overline{w})$ be the embedding function with the tight decomposition function for $\dot{x} = f(x,w)$. For any $\eta \in \mathbb{R}^n_{\geq 0}$

$$\mu_{\infty,\eta}\left(\frac{\partial f}{\partial x}(x,w)\right) \leq c \quad \Longleftrightarrow \quad \mu_{\infty,\eta\otimes I_2}\left(\frac{\partial e}{\partial\left[\frac{\underline{x}}{\overline{x}}\right]}(\underline{x},\overline{x},\underline{w},\overline{w})\right) \leq c$$

[4] Jafarpour and Coogan, "Monotoncity and contraction on polyhedral cones", TAC, 2024

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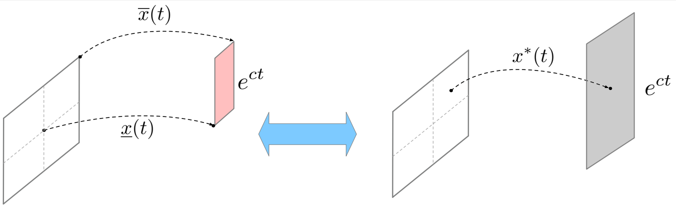
Consequence 1: hyper-rectangles evolve with ℓ_∞ contraction rate of original system



Mixed Monotone hyper-rectangle   Contraction tube

[4] Jafarpour and Coogan, "Monotoncity and contraction on polyhedral cones", TAC, 2024

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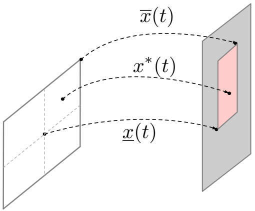
Consequence 2: Mixed Monotone is sharper than contraction wrt to ℓ_∞



Gray = contraction tube
Red = Mixed Monotone hyper-rectangle

$$\|x^*(t) - \underline{x}(t)\|_{\infty,\eta} \leq e^{ct}\|x^*(0) - \underline{x}(0)\|_{\infty,\eta}$$
$$\|x^*(t) - \overline{x}(t)\|_{\infty,\eta} \leq e^{ct}\|x^*(0) - \overline{x}(0)\|_{\infty,\eta}$$

[4] Jafarpour and Coogan, "Monotoncity and contraction on polyhedral cones", TAC, 2024

Summary

- we introduced mixed monotone theory, which constructs an embedding system for reachability analysis
- we identified the tightest possible embedding system for this approach.
- we showed that the rate of contraction (with respect to diagonal ℓ_∞-norms) of the tightest embedding system matches that of the original system.

Future Research

- mixed monotone theory with respect to polyhedral cones (with Sam Coogan)
- contraction-based and mixed monotone reachability for stochastic dynamical system (with Yongxin Chen)

SJ and Z. Liu and Y. Chen. Probabilistic Reachability Analysis of Stochastic Control Systems. arXiv, 2024 (`https://arxiv.org/pdf/2407.12225v2`)