Mixed-monotone Theory for Verification of Autonomous System

Saber Jafarpour

University of Colorado **Boulder**

September 11, 2024

Energy/power systems

Air mobility

Autonomous driving

Manufacturing

Transportation systems

Agriculture

Safety-critical Autonomous Systems

Introduction

Energy/power systems

Air mobility

Autonomous driving

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An important goal (Safe Autonomy)

Perform their tasks while ensuring **safety** and **robustness** of the system.

In this talk: Autonomous systems with learning-enabled components

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Machine learning is a driving forces for developments in autonomous systems

In this talk: Autonomous systems with learning-enabled components

Machine learning is a driving forces for developments in autonomous systems

- availability of data and computation tools
- performance and efficiency

Learning-enabled Autonomous Systems

Motivations and Success Stories

In this talk: Autonomous systems with learning-enabled components

Machine learning is a driving forces for developments in autonomous systems

- availability of data and computation tools
- performance and efficiency

Success stories and potential applications

NVIDIA self driving car

Amazon fulfillment centers

Manufacturing

Learning-enabled Autonomous Systems

Safety Assurance as a Challenge

But can we ensure their safety?

Tesla Slams Right Into Overturned Truck While on Autopilot

Waymo driverless car strikes bicyclist in San Francisco, causes minor injuries

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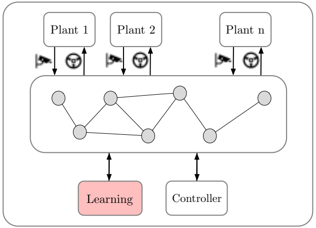
What is different with Learning-based components?



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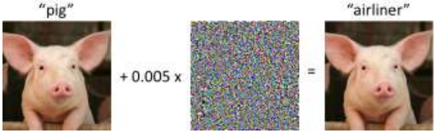

Image credit: MIT CSAIL

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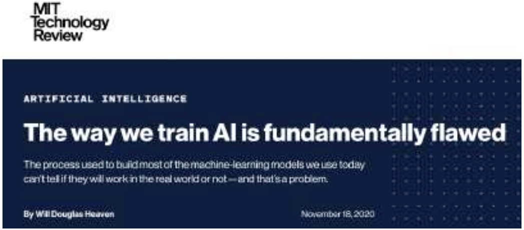
MIT Technology Review

ARTIFICIAL INTELLIGENCE

The way we train AI is fundamentally flawed

The process used to build most of the machine-learning models we use today can't tell if they will work in the real world or not—and that's a problem.

By Will Douglas Heaven

November 18, 2020

But can we ensure their safety?

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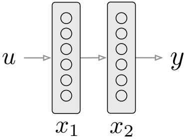
- limited guarantee in their design
- large # of parameters with nonlinearity



$478 \times 100 \times 100 \times 10$

$\#$ of parameters ~ 90000

$\#$ of activation patterns $\sim 10^{60}$

Learning-enabled Autonomous Systems

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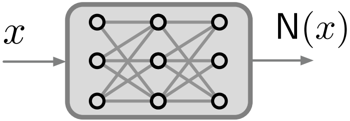
Rigorous and **computationally efficient** methods for safety assurance

ML focus on safety and robustness of **stand-alone** learning algorithms



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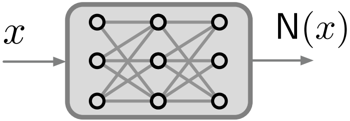

Different approaches:

- analysis (Goodfellow et al., 2015, Zhang et al., 2019, Fazlyab et al., 2023)
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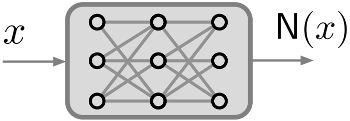

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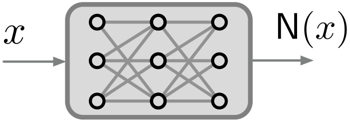

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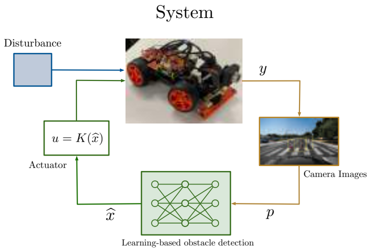
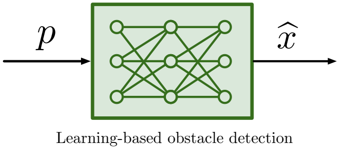
New challenges arises when learning algorithms are used **in-the-loop**

Perception-based Obstacle Avoidance



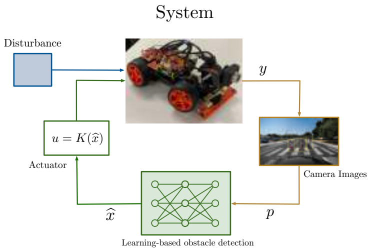
In-the-loop

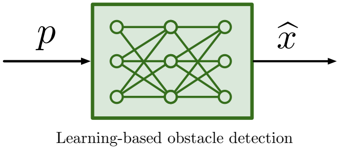


trained offline using images

Stand-alone

Perception-based Obstacle Avoidance





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In-the-loop

Stand-alone

- **stand-alone**: estimation of states using learning algorithm
- **in-the-loop**: closed-loop system avoid the obstacle

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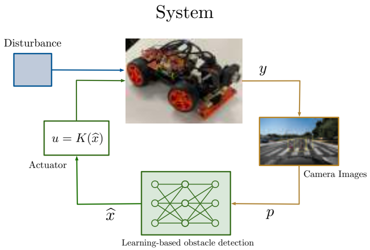
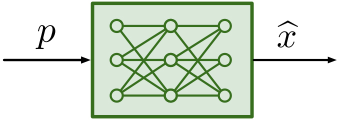


In-the-loop



Learning-based obstacle detection

trained offline using images

Stand-alone

- **stand-alone**: estimation of states using learning algorithm
- **in-the-loop**: closed-loop system avoid the obstacle

> **In-the-loop**: how the autonomous system perform with the learning algorithm as a part of it.

Ensure safety of the autonomous system with learning algorithms **in-the-loop**

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Safety of autonomous system using **reachability analysis**

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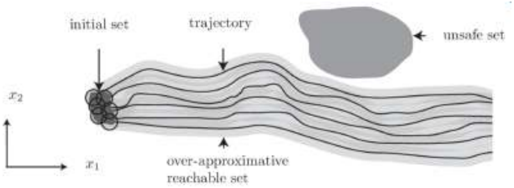
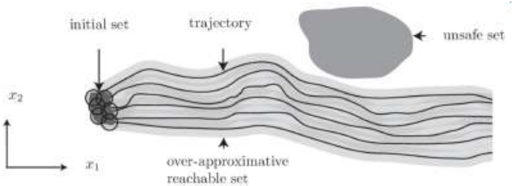
Safety of autonomous system using **reachability analysis**



Reachability analysis estimates the evolution of the autonomous system

Ensure safety of the autonomous system with learning algorithms **in-the-loop**

Safety of autonomous system using **reachability analysis**



Reachability analysis estimates the evolution of the autonomous system

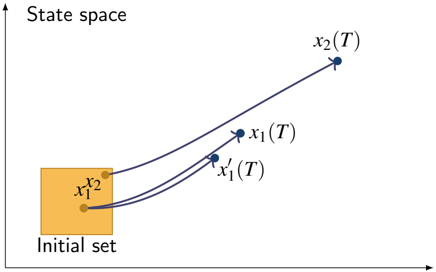
In this talk:

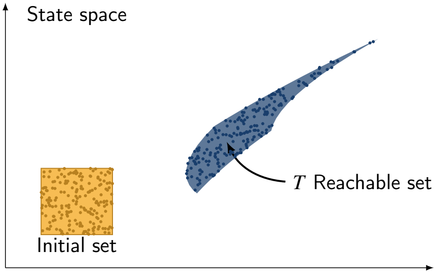
1. control-theoretic tools for efficient and scalable reachability
2. applications to safety assurance of learning-enabled systems

Outline of this talk

- **Reachability Analysis**
- Monotone System Theory
- Neural Network Controlled Systems

System : $\dot{x} = f(x, w)$ **State** : $x \in \mathbb{R}^n$ **Uncertainty** : $w \in \mathcal{W} \subseteq \mathbb{R}^m$





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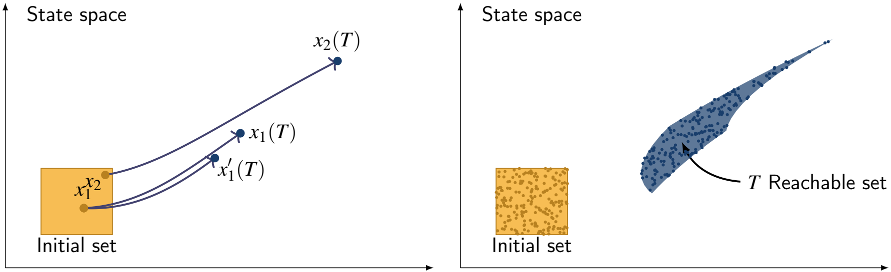

What are the possible states of the system at time T?

- T**-reachable sets** characterize evolution of the system

$$\mathcal{R}_f(T, \mathcal{X}_0, \mathcal{W}) = \{x_w(T) \mid x_w(\cdot) \text{ is a traj for some } w(\cdot) \in \mathcal{W} \text{ with } x_0 \in \mathcal{X}_0\}$$

A large number of **safety specifications** can be represented using T-reachable sets

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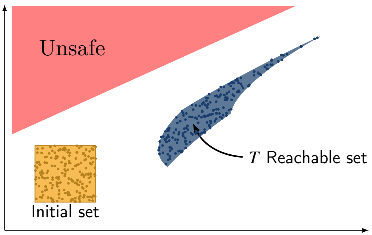
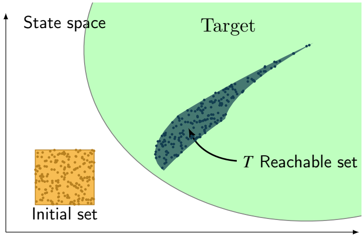
- Example: Reach-avoid problem





$$\mathcal{R}_f(T, \mathcal{X}_0, \mathcal{W}) \cap \text{ Unsafe set } = \emptyset$$

$$\mathcal{R}_f(T_{\text{final}}, \mathcal{X}_0, \mathcal{W}) \subseteq \text{Target set}$$

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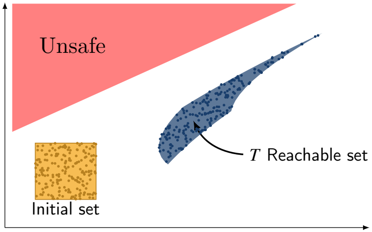

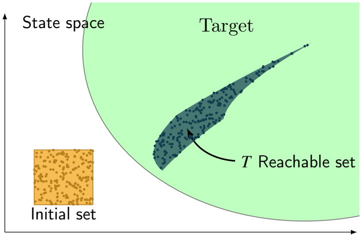

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Combining different instantiation of Reach-avoid problem $\Longrightarrow$
diverse range of specifications
(complex planning using logics, invariance, stability)

Computing the T-reachable sets are computationally challenging

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Solution: over-approximations of reachable sets

Over-approximation: $\mathcal{R}_f(T, \mathcal{X}_0, \mathcal{W}) \subseteq \overline{\mathcal{R}}_f(T, \mathcal{X}_0, \mathcal{W})$

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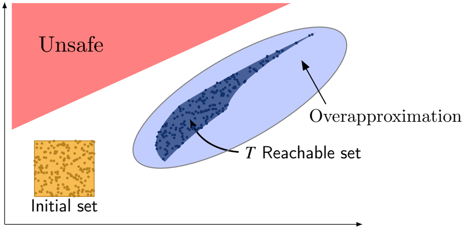

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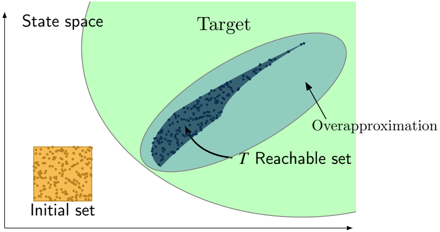

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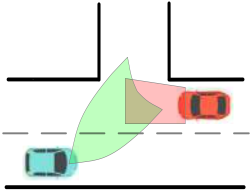
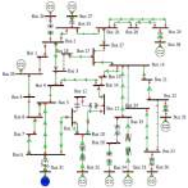
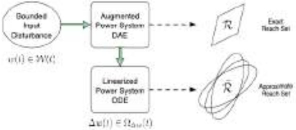
Autonomous Driving:



Althoff, 2014

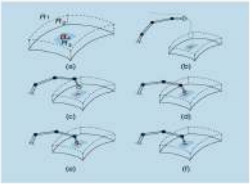
Power grids:



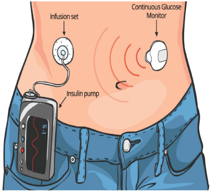


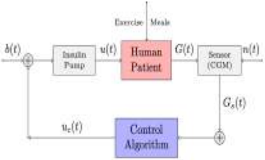
Chen and Domınguez-Garcıa, 2016

Robot-assisted Surgery:



Drug Delivery:





Chen, Dutta, and Sankaranarayanan, 2017

Reachability of dynamical system is an old problem: $\sim$ 1980

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Different approaches for approximating reachable sets

- Linear, and piecewise linear systems (Ellipsoidal methods) (Kurzhanski and Varaiya, 2000)
- Optimization-based approaches (Hamilton-Jacobi, Level-set method) (Bansal et al., 2017, Mitchell et al., 2002, Herbert et al., 2021)
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In this talk: use control-theoretic tools to develop scalable and computationally efficient approaches for reachability

- Reachability Analysis
- **Monotone System Theory**
- Neural Network Controlled Systems

A dynamical system $\dot{x} = f(x, w)$ is monotone if

$$x_u(0) \leq y_w(0) \quad \text{and} \quad u \leq w \quad \Longrightarrow \quad x_u(t) \leq y_w(t) \quad \text{for all time}$$

where $\leq$ is the component-wise partial order.

[1] Angeli and Sontag, "Monotone control systems", IEEE TAC, 2003

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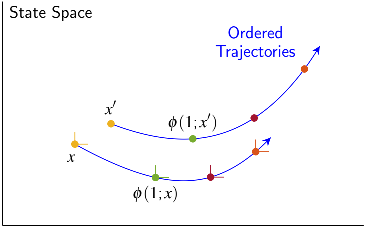
Theorem[1]: Monotonicity test

1. $\frac{\partial f}{\partial x}(x, w)$ is Metzler (off-diag ≥ 0)
2. $\frac{\partial f}{\partial w}(x, w) \geq 0$



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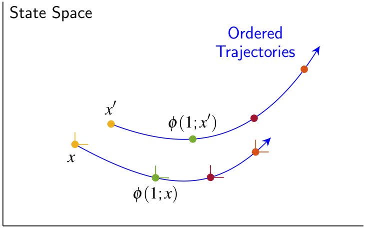

In this talk: monotone system theory for reachability analysis

[1] Angeli and Sontag, "Monotone control systems", IEEE TAC, 2003

Monotone System

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_1 + w \\ x_1 \end{bmatrix}$$

Non-monotone System

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_2 + w \\ x_1 \end{bmatrix}$$

Theorem (classical result)

For a monotone system $\dot{x} = f(x, w)$ with $w \in \mathcal{W} = [\underline{w}, \overline{w}]$

$$\mathcal{R}_f(t, [\underline{x}_0, \overline{x}_0], [\underline{w}, \overline{w}]) \subseteq [x_{\underline{w}}(t), x_{\overline{w}}(t)]$$

where $x_{\underline{w}}(\cdot)$ (resp. $x_{\overline{w}}(\cdot)$) is the trajectory with disturbance $\underline{w}$ (resp. $\overline{w}$) starting at $\underline{x}_0$ (resp. $\overline{x}_0$)

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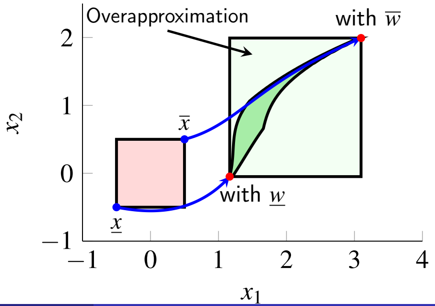
Example:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_1 + w \\ x_1 \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$



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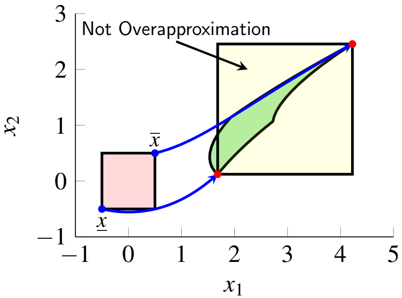

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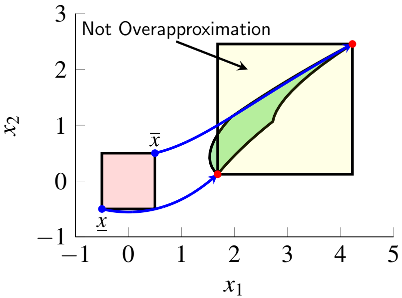

How to over-approximate the reachable sets of non-monotone systems?

- **Key idea:** embed the dynamical system on $\mathbb{R}^n$ into a dynamical system on $\mathbb{R}^{2n}$
- Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$

Original system

$$\dot{x} = f(x, w)$$

Embedding system

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}),$$
$$\dot{\overline{x}} = \overline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$$

$\underline{d}, \overline{d}$ are **decomposition functions** s.t.

1. $f(x, w) = \underline{d}(x, x, w, w)$ for every x, w
2. cooperative: $(\underline{x}, \underline{w}) \mapsto \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$
3. competitive: $(\overline{x}, \overline{w}) \mapsto \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$
4. the same properties for $\overline{d}$

- **Key idea:** embed the dynamical system on $\mathbb{R}^n$ into a dynamical system on $\mathbb{R}^{2n}$
- Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$

Original system

$$\dot{x} = f(x, w)$$

Embedding system

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}),$$
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4. the same properties for $\overline{d}$

f locally Lipschitz $\implies$ a decomposition function exists

Southeast partial order $\leq_{\mathrm{SE}}$:

$$\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} y \\ \widehat{y} \end{bmatrix} \quad \Longleftrightarrow \quad x \leq y \quad \text{and} \quad \widehat{y} \leq \widehat{x}$$

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Theorem (Classical Result)

The embedding system is a monotone dynamical system on $\mathbb{R}^{2n}$ with respect to the **southeast** partial order $\leq_{\mathrm{SE}}$:

$$\begin{bmatrix} \underline{x}_0 \\ \overline{x}_0 \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} \underline{y}_0 \\ \overline{y}_0 \end{bmatrix}, \quad \begin{bmatrix} \underline{u} \\ \overline{u} \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} \underline{w} \\ \overline{w} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} \underline{x}_{[\underline{u},\overline{u}]}(t) \\ \overline{x}_{[\underline{u},\overline{u}]}(t) \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} \underline{y}_{[\underline{w},\overline{w}]}(t) \\ \overline{y}_{[\underline{w},\overline{w}]}(t) \end{bmatrix}$$

Southeast partial order $\leq_{\mathrm{SE}}$:

$$\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} y \\ \widehat{y} \end{bmatrix} \iff x \leq y \quad \text{and} \quad \widehat{y} \leq \widehat{x}$$

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Key idea: use monotonicity of the embedding system to study the original dynamical system

A short (and incomplete) Literature review:

J-L. Gouze and L. P. Hadeler. Monotone flows and order intervals. Nonlinear World, 1994

G. Enciso, H. Smith, and E. Sontag. Nonmonotone systems decomposable into monotone systems with negative feedback . Journal of Differential Equations, 2006.

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In this talk: use embedding system to study reachability of the original system

Original System:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_2 + w \\ x_1 \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3]$$

blue = cooperative, red = competitive

Decomposition function

$$\underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \begin{bmatrix} \underline{x}_2^3 + \underline{w} \\ \underline{x}_1 \end{bmatrix} + \begin{bmatrix} -\overline{x}_2 \\ 0 \end{bmatrix}$$

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Decomposition function

$$\underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \begin{bmatrix} \underline{x}_2^3 + \underline{w} \\ \underline{x}_1 \end{bmatrix} + \begin{bmatrix} -\overline{x}_2 \\ 0 \end{bmatrix}$$

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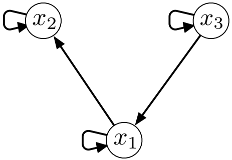
- **Metzler/non-Metzler** decomposition: $A = [A]^{\mathrm{Mzl}} + [A]^{\mathrm{n-Mzl}}$
- Example: $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies [A]^{\mathrm{Mzl}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [A]^{\mathrm{n-Mzl}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Linear systems

Original system

$$\dot{x} = Ax + Bw$$



Embedding system

$$\dot{\underline{x}} = [A]^{\mathrm{Mzl}}\underline{x} + [A]^{\mathrm{n-Mzl}}\overline{x} + B^{+}\underline{w} + B^{-}\overline{w}$$
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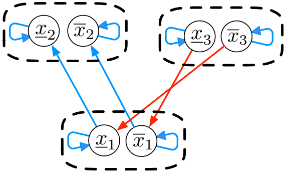


How to compute a decomposition function for a system?

- Assume $f : \mathbb{R} \to \mathbb{R}$ is scalar-valued:

Mean-value Inequality

$$f(\underline{x}) + \left[\min_{z\in[\underline{x},\overline{x}]} \frac{\partial f}{\partial x}\right](\overline{x} - \underline{x}) \leq f(x) \leq f(\underline{x}) + \left[\max_{z\in[\underline{x},\overline{x}]} \frac{\partial f}{\partial x}\right](\overline{x} - \underline{x})$$

Then

$$\begin{bmatrix} \underline{d}(\underline{x},\overline{x}) \\ \overline{d}(\underline{x},\overline{x}) \end{bmatrix} = \begin{bmatrix} \left[\min_{z\in[\underline{x},\overline{x}]} \frac{\partial f}{\partial x}\right]^{+} & \left[\min_{z\in[\underline{x},\overline{x}]} \frac{\partial f}{\partial x}\right]^{-} \\ \left[\max_{z\in[\underline{x},\overline{x}]} \frac{\partial f}{\partial x}\right]^{-} & \left[\max_{z\in[\underline{x},\overline{x}]} \frac{\partial f}{\partial x}\right]^{+} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix}$$

where $[A]^{+} = \max\{A, 0\}$ and $[A]^{-} = \min\{A, 0\}$.

How to compute a decomposition function for a system?

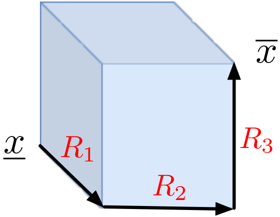
Theorem[2]

Jacobian-based: $\dot{x} = f(x,w)$ with differentiable f, then

$$\begin{bmatrix} \underline{d}(\underline{x},\overline{x},\underline{u},\overline{u}) \\ \overline{d}(\underline{x},\overline{x},\underline{w},\overline{w}) \end{bmatrix} = \begin{bmatrix} [\underline{A}]^+ & [\underline{A}]^- \\ [\overline{A}]^- & [\overline{A}]^+ \end{bmatrix} \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix} + \begin{bmatrix} [\underline{B}]^+ & [\underline{B}]^- \\ [\overline{B}]^- & [\overline{B}]^+ \end{bmatrix} \begin{bmatrix} \underline{w} \\ \overline{w} \end{bmatrix} + \begin{bmatrix} f(\underline{x},\underline{w}) \\ f(\underline{x},\underline{w}) \end{bmatrix}$$

$\underline{x} \mapsto R_1 \mapsto R_2 \mapsto \ldots \mapsto R_n \mapsto \overline{x}$, then the i-th column of $\underline{A}$ is $\min_{z\in R_i, u\in[\underline{w},\overline{w}]} \frac{\partial f_i}{\partial x}(z,u)$

- Interval analysis for computing Jacobian bounds.
- `immrax`: Toolbox that implements interval analysis in `JAX`.



[2] **SJ** and A. Harapanahalli and S. Coogan, L4DC, 2023

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Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ and

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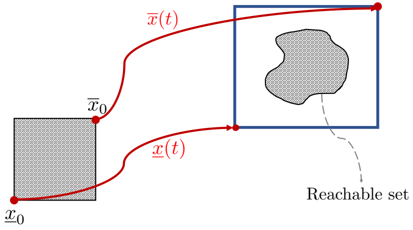
Then $\mathcal{R}_f(t, \mathcal{X}_0, \mathcal{W}) \subseteq [\underline{x}(t), \overline{x}(t)]$



[3] H. Smith, Journal of Difference Equations and Applications, 2008

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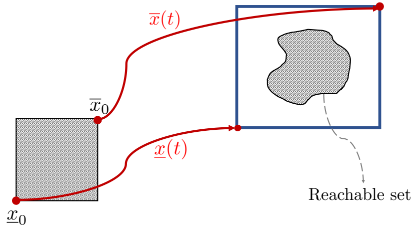

a single trajectory of embedding system provides **lower bound** ($\underline{x}$) and **upper bound** ($\overline{x}$) for the trajectories of the original system.

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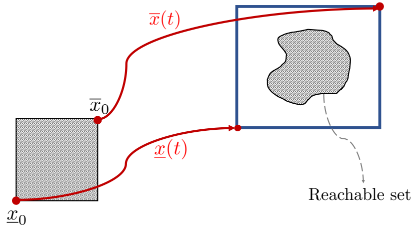

a single trajectory of embedding system provides **lower bound** ($\underline{x}$) and **upper bound** ($\overline{x}$) for the trajectories of the original system.

(Computational efficient): solve for one trajectory of embedding system

(Scalable): embedding system is $2n$-dimensional

[3]H. Smith, Journal of Difference Equations and Applications, 2008

Original System:

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$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$

blue = cooperative, red = competitive

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$$\begin{bmatrix} \underline{x}_1(0) \\ \underline{x}_2(0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \qquad \begin{bmatrix} \overline{x}_1(0) \\ \overline{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

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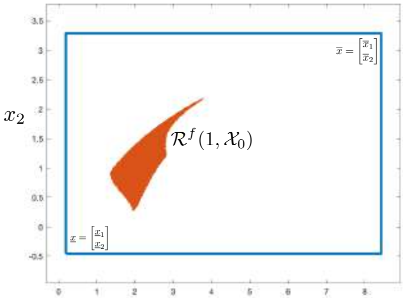

- Reachability Analysis
- Monotone System Theory
- **Neural Network Controlled Systems**

- **In this part:** Learning-based component as a controller

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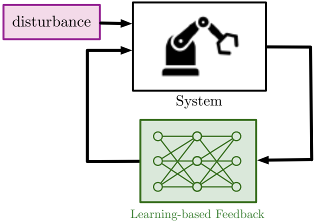

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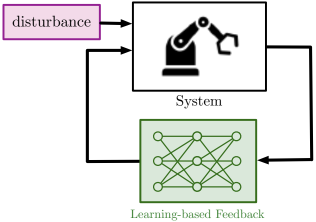
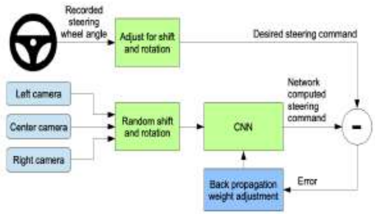


Issues with traditional controllers:

1. computationally burdensome
2. interaction with human
3. complicated representation

Self driving vehicles:

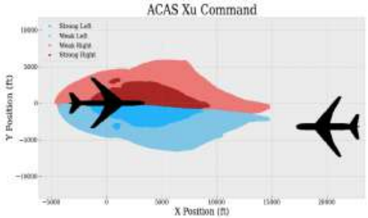


M. Bojarski, et al., NeurIPS, 2016.

Robotic motion planning:

M. Everett, et. al., IROS, 2018.

Collision avoidance:



K. Julian, et. al., DASC, 2016.

Safety of learning-enabled autonomous systems **cannot be completely ensured** at the design level[4]

[4]Institute for Defense Analysis, The Status of Test, Evaluation, Verification, and Validation of Autonomous Systems, 2018

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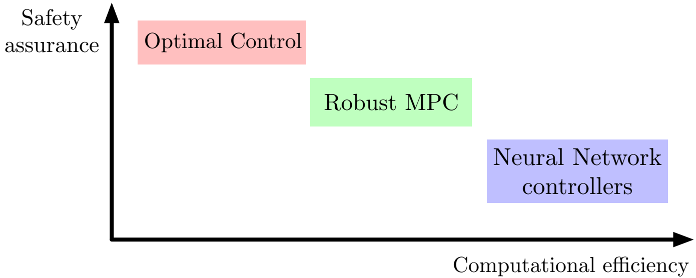

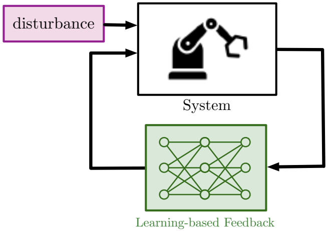

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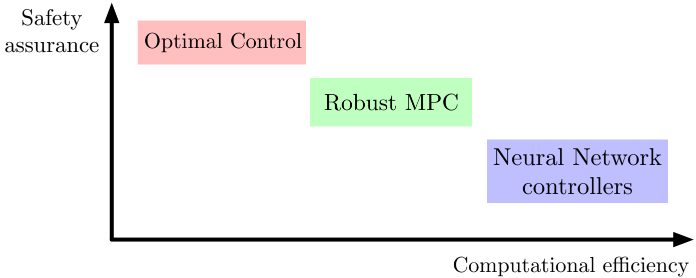

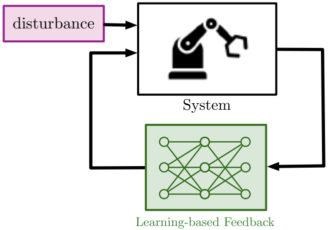

Design a mechanism that can do **run-time** safety verification

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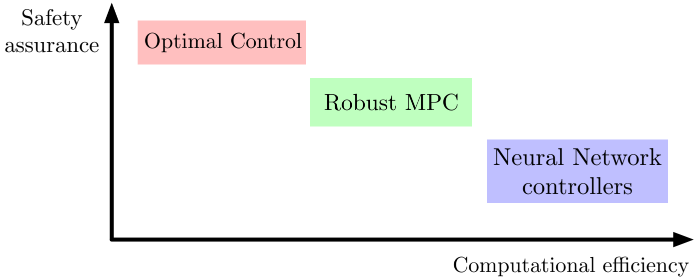

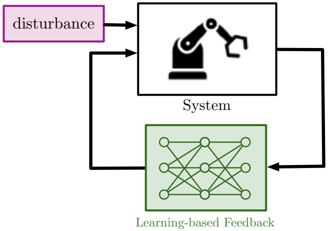

Design a mechanism that can do **run-time** safety verification

Our approach: reachable set over-approximations for some time in future.

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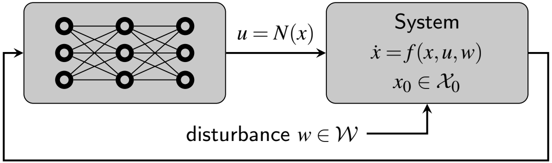
An open-loop nonlinear system with a neural network controller

$$\dot{x} = f(x, u, w),$$
$$u = N(x),$$

safety of the closed-loop system

$$\dot{x} = f(x, N(x), w) := f^c(x, w)$$



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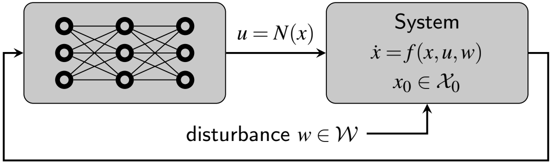

$u = N(x)$ is **pre-trained** feed-forward neural network with k-layer:

$$\xi^{(i)}(x) = \phi^{(i)}(W^{(i-1)}\xi^{(i-1)}(x) + b^{(i-1)})$$
$$x = \xi^{(0)}, \quad u = W^{(k)}\xi^{(k)}(x) + b^{(k)} := N(x),$$

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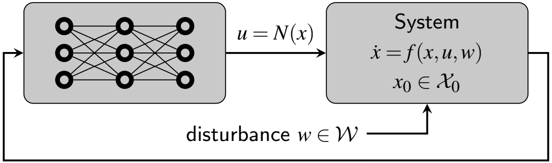

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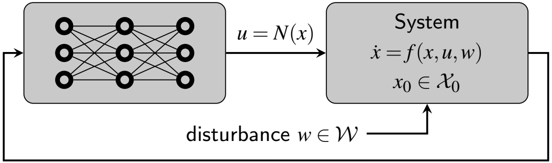
directly performing reachability on f^c is computationally challenging

An open-loop nonlinear system with a neural network controller

$$\dot{x} = f(x, u, w),$$
$$u = N(x),$$

safety of the closed-loop system

$$\dot{x} = f(x, N(x), w) := f^c(x, w)$$



$u = N(x)$ is **pre-trained** feed-forward neural network with k-layer:

$$\xi^{(i)}(x) = \phi^{(i)}(W^{(i-1)}\xi^{(i-1)}(x) + b^{(i-1)})$$
$$x = \xi^{(0)}, \quad u = W^{(k)}\xi^{(k)}(x) + b^{(k)} := N(x),$$

Rigorousness of control tools + effectiveness of ML tools

Combine our reachability frameworks with neural network verification methods

Input-output bounds: Given a neural network controller $u = N(x)$

$$\underline{u}_{[\underline{x},\overline{x}]} \leq N(x) \leq \overline{u}_{[\underline{x},\overline{x}]}, \qquad \text{for all } x \in [\underline{x}, \overline{x}]$$

[5] H. Zhang et al., NeurIPS 2018.

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Many neural network verification algorithms can produce these bounds.

ex. CROWN (H. Zhang et al., 2018), LipSDP (M. Fazlyab et al., 2019), IBP (S. Gowal et al., 2018).

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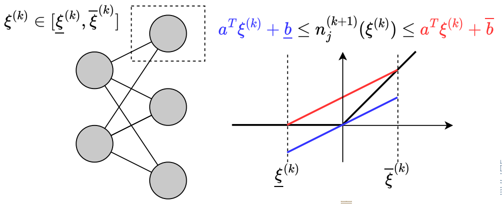
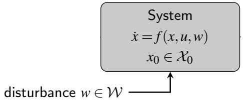
CROWN[5]

- Bounding the value of each neurons
- Linear upper and lower bounds on the activation function

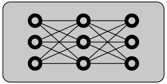


[5]H. Zhang et al., NeurIPS 2018.

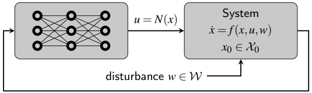
Reachability of open-loop system treating u as a parameter



Neural network verification algorithm for bounds on u



Reachability of open-loop system + Neural network verification bounds



$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{u}, \overline{u}, \underline{w}, \overline{w})$$
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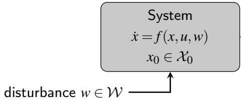

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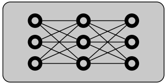

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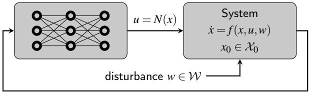

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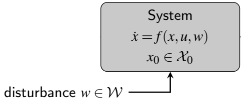

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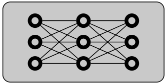

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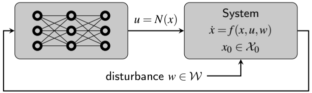

Composition approach over-approximation:

$$\mathcal{R}_{f^c}(t, \mathcal{X}_0, \mathcal{W}) \subseteq [\underline{x}(t), \overline{x}(t)]$$

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{u}, \overline{u}, \underline{w}, \overline{w})$$
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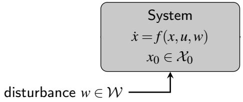

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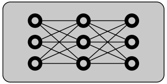

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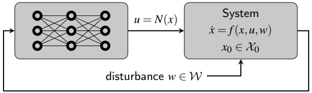


Composition approach over-approximation:
$$\mathcal{R}_{f^c}(t, \mathcal{X}_0, \mathcal{W}) \subseteq [\underline{x}(t), \overline{x}(t)]$$

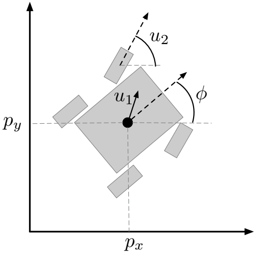
It lead to overly-conservative estimates of reachable set

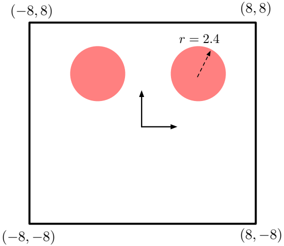
Dynamics of bicycle

$$\dot{p_x} = v\cos(\phi + \beta(u_2)) \qquad \dot{\phi} = \frac{v}{\ell_r}\sin(\beta(u_2))$$

$$\dot{p_y} = v\sin(\phi + \beta(u_2)) \qquad \dot{v} = u_1$$

$$\beta(u_2) = \arctan\left(\frac{l_r}{l_f + l_r}\tan(u_2)\right)$$





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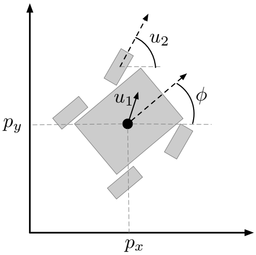

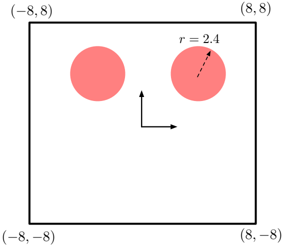

Goal: steer the bicycle to the origin avoiding the obstacles

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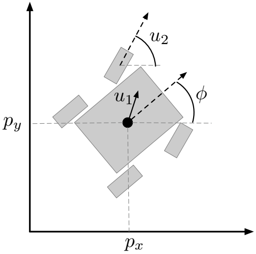
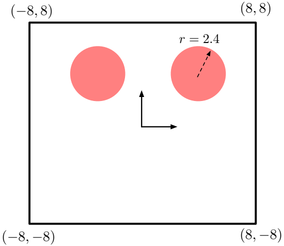




Goal: steer the bicycle to the origin avoiding the obstacles

- train a feedforward neural network $4 \mapsto 100 \mapsto 100 \mapsto 2$ using data from model predictive control

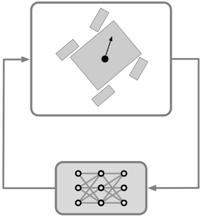
Reachability of Closed-loop System

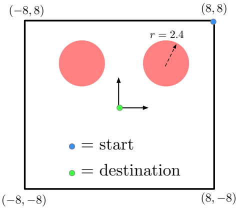
Case Study: Bicycle Model

- start from $(8, 8)$ toward $(0, 0)$
- $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ with

$$\underline{x}_0 = \begin{pmatrix} 7.95 & 7.95 & -\frac{\pi}{3} - 0.01 & 1.99 \end{pmatrix}^\top$$
$$\overline{x}_0 = \begin{pmatrix} 8.05 & 8.05 & -\frac{\pi}{3} + 0.01 & 2.01 \end{pmatrix}^\top$$

- CROWN for verification of neural network





Embedding system:

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{u}, \overline{u}, \underline{w}, \overline{w})$$
$$\dot{\overline{x}} = \overline{d}(\underline{x}, \overline{x}, \underline{u}, \overline{u}, \underline{w}, \overline{w})$$

$\underline{u} \leq N(x) \leq \overline{u}$, for every $x \in [\underline{x}, \overline{x}]$.

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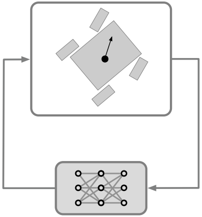
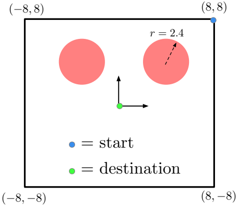
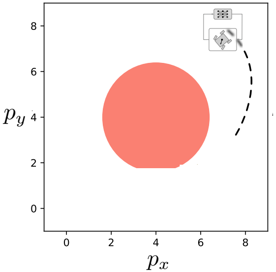


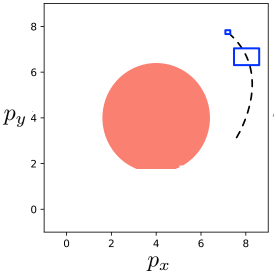


Euler integration with step h:

$$\underline{x}_1 = \underline{x}_0 + h\underline{d}(\underline{x}_0, \overline{x}_0, \underline{u}_0, \overline{u}_0, \underline{w}, \overline{w})$$
$$\overline{x}_1 = \overline{x}_0 + h\overline{d}(\underline{x}_0, \overline{x}_0, \underline{u}_0, \overline{u}_0, \underline{w}, \overline{w})$$

$\underline{u}_0 \leq N(x) \leq \overline{u}_0$, for every $x \in [\underline{x}_0, \overline{x}_0]$.





- start from $(8,8)$ toward $(0,0)$
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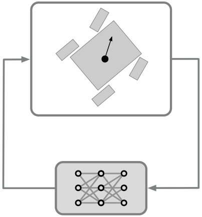
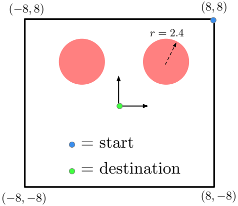
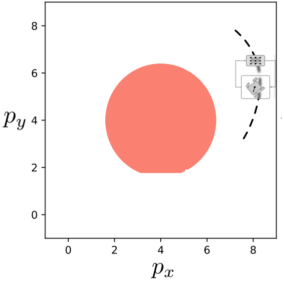
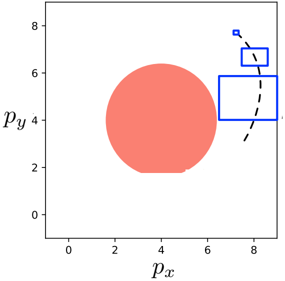




Euler integration with step h:

$$\underline{x}_2 = \underline{x}_1 + h\underline{d}(\underline{x}_1, \overline{x}_1, \underline{u}_1, \overline{u}_1, \underline{w}, \overline{w})$$
$$\overline{x}_2 = \overline{x}_1 + h\overline{d}(\underline{x}_1, \overline{x}_1, \underline{u}_1, \overline{u}_1, \underline{w}, \overline{w})$$





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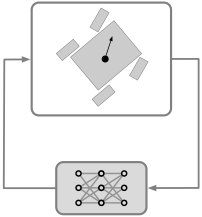

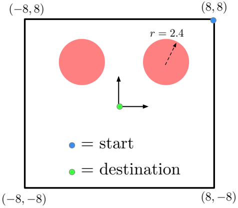

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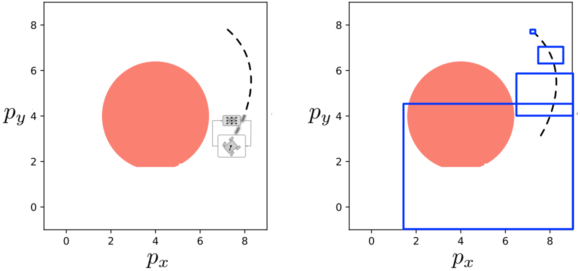

Stabilizing Effect of Neural Network Controllers

Issues with the compositional approach

Neural network controller as **disturbances** (worst-case scenario)
It does not capture the **stabilizing** effect of the neural network.

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It does not capture the **stabilizing** effect of the neural network.

An illustrative example

$\dot{x} = x + u + w$ with controller $u = -Kx$, for some unknown $1 < K \leq 3$.

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It does not capture the **stabilizing** effect of the neural network.

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$\dot{x} = x + u + w$ with controller $u = -Kx$, for some unknown $1 < K \leq 3$.

Compositional approach

First find the bounds $\underline{u} \leq Kx \leq \overline{u}$, then

$$\dot{\underline{x}} = \underline{x} + \underline{u} + \underline{w}$$
$$\dot{\overline{x}} = \overline{x} + \overline{u} + \overline{w}$$

This system is unstable.

Interaction-aware approach

First replace $u = Kx$ in the system, then

$$\dot{\underline{x}} = (1 - K)\underline{x} + \underline{w}$$
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This system is stable.

We need to know the **functional** dependencies of neural network bounds

Functional bounds: Given a neural network controller $u = N(x)$

$$\underline{N}_{[\underline{x},\overline{x}]}(x) \leq N(x) \leq \overline{N}_{[\underline{x},\overline{x}]}(x), \quad \text{for all } x \in [\underline{x}, \overline{x}]$$

[6] H. Zhang et al., NeurIPS 2018.

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- Example: CROWN[6] can provide functional bounds.

CROWN functional bounds:

$$\begin{aligned}\underline{N}_{[\underline{x},\overline{x}]}(x) &= \underline{A}_{[\underline{x},\overline{x}]}x + \underline{b}_{[\underline{x},\overline{x}]},\\ \overline{N}_{[\underline{x},\overline{x}]}(x) &= \overline{A}_{[\underline{x},\overline{x}]}x + \overline{b}_{[\underline{x},\overline{x}]}\end{aligned}$$

CROWN input-output bounds:

$$\begin{aligned}\underline{u}_{[\underline{x},\overline{x}]} &= \underline{A}^{+}_{[\underline{x},\overline{x}]}\overline{x} + \overline{A}^{-}_{[\underline{x},\overline{x}]}\underline{x} + \underline{b}_{[\underline{x},\overline{x}]},\\ \overline{u}_{[\underline{x},\overline{x}]} &= \overline{A}^{+}_{[\underline{x},\overline{x}]}\overline{x} + \underline{A}^{-}_{[\underline{x},\overline{x}]}\underline{x} + \overline{b}_{[\underline{x},\overline{x}]}\end{aligned}$$

[6] H. Zhang et al., NeurIPS 2018.

Theorem[7]

Original system

$$\dot{x} = f(x, N(x), w)$$

Embedding system

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\overline{x}} \end{bmatrix} = \begin{bmatrix} [\underline{H}]^+ - \underline{J}_{[\underline{x},\overline{x}]} & [\underline{H}]^- \\ [\overline{H}]^+ - \overline{J}_{[\underline{x},\overline{x}]} & [\overline{H}]^- \end{bmatrix} \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix} + \begin{bmatrix} -[\underline{J}_{[\underline{w},\overline{w}]}]^- & [\underline{J}_{[\underline{w},\overline{w}]}]^+ \\ -[\overline{J}_{[\underline{w},\overline{w}]}]^- & [\overline{J}_{[\underline{w},\overline{w}]}]^+ \end{bmatrix} \begin{bmatrix} \underline{w} \\ \overline{w} \end{bmatrix} + Q$$

$\underline{H}$ and $\overline{H}$ capture the effect of interactions between nonlinear system and neural network.

Interaction-aware over-approximation:

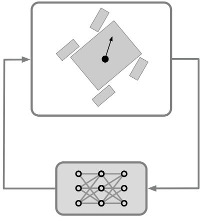
$$\mathcal{R}_{f^c}(t, \mathcal{X}_0, \mathcal{W}) \subseteq [\underline{x}(t), \overline{x}(t)]$$

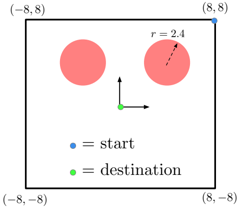
[7] **SJ** and A. Harapanahalli and S. Coogan, under review, 2023

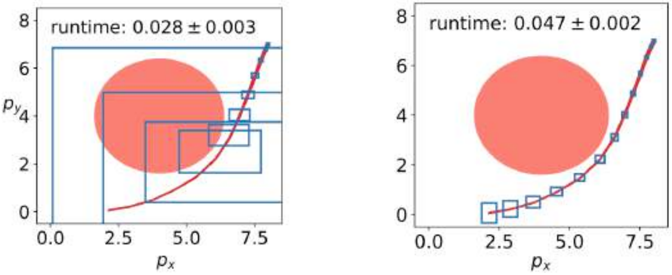
- start from $(8,7)$ toward $(0,0)$
- $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ with

$$\underline{x}_0 = \begin{pmatrix} 7.95 & 6.95 & -\frac{2\pi}{3} - 0.01 & 1.99 \end{pmatrix}^\top$$
$$\overline{x}_0 = \begin{pmatrix} 8.05 & 7.05 & -\frac{2\pi}{3} + 0.01 & 2.01 \end{pmatrix}^\top$$

- CROWN for verification of neural network





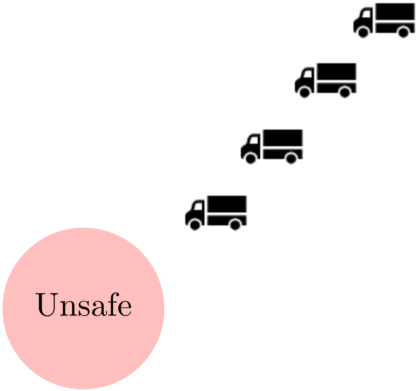


Composition approach   Interaction-aware approach

Dynamics of the jth vehicle

$$\dot{p}_x^j = v_x^j, \qquad \dot{v}_x^j = \tanh(u_x^j) + w_x^j,$$
$$\dot{p}_y^j = v_y^j, \qquad \dot{v}_y^j = \tanh(u_y^j) + w_y^j,$$

where $w_x^j, w_y^j \sim \mathcal{U}([-0.001, 0.001])$.



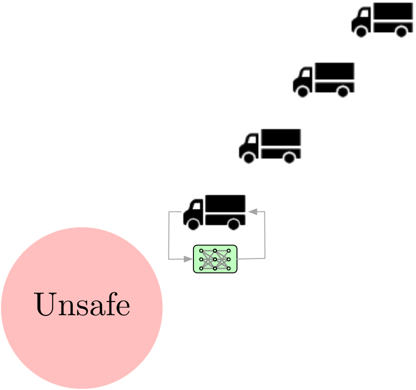
Dynamics of the jth vehicle

$$\dot{p}_x^j = v_x^j, \qquad \dot{v}_x^j = \tanh(u_x^j) + w_x^j,$$
$$\dot{p}_y^j = v_y^j, \qquad \dot{v}_y^j = \tanh(u_y^j) + w_y^j,$$

where $w_x^j, w_y^j \sim \mathcal{U}([-0.001, 0.001])$. First vehicle uses a neural network controller

$4 \times 100 \times 100 \times 2$, with ReLU activations

and is trained using trajectory data from an MPC controller for the first vehicle.



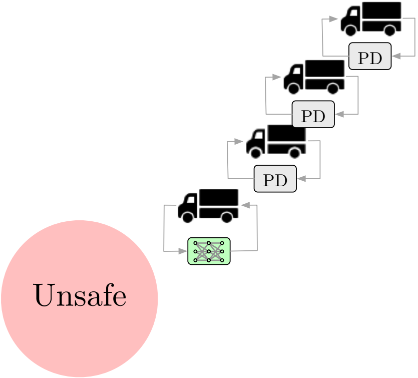
Dynamics of the jth vehicle

$$\dot{p}_x^j = v_x^j, \qquad \dot{v}_x^j = \tanh(u_x^j) + w_x^j,$$
$$\dot{p}_y^j = v_y^j, \qquad \dot{v}_y^j = \tanh(u_y^j) + w_y^j,$$

where $w_x^j, w_y^j \sim \mathcal{U}([-0.001, 0.001])$. Other vehicles use PD controller

$$u_d^j = k_p\left(p_d^{j-1} - p_d^j - r\frac{v_d^{j-1}}{\|v^{j-1}\|_2}\right) + k_v(v_d^{j-1} - v_d^j),$$

where $d \in \{x, y\}$.

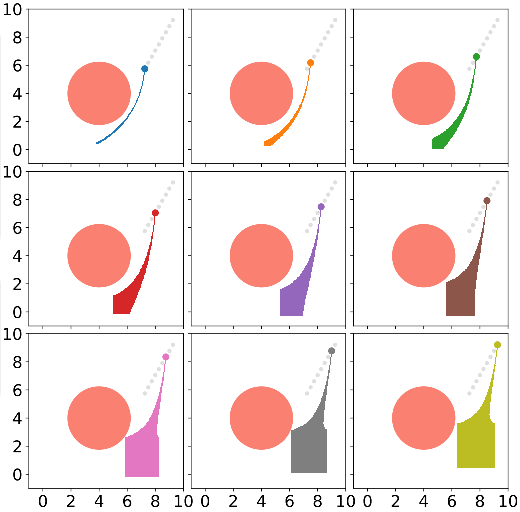


Dynamics of the jth vehicle

$$\dot{p}_x^j = v_x^j, \qquad \dot{v}_x^j = \tanh(u_x^j) + w_x^j,$$
$$\dot{p}_y^j = v_y^j, \qquad \dot{v}_y^j = \tanh(u_y^j) + w_y^j,$$

where $w_x^j, w_y^j \sim \mathcal{U}([-0.001, 0.001])$.

- compositional approach
- a platoon of 9 vehicles
- reachable overapproximations for $t \in [0, 1.5]$



Dynamics of the jth vehicle

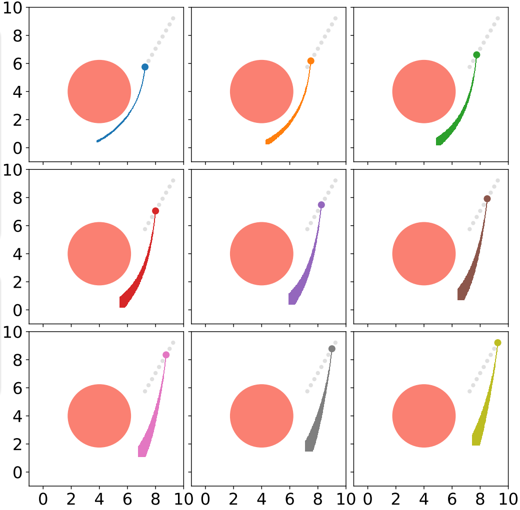
$$\dot{p}_x^j = v_x^j, \qquad \dot{v}_x^j = \tanh(u_x^j) + w_x^j,$$
$$\dot{p}_y^j = v_y^j, \qquad \dot{v}_y^j = \tanh(u_y^j) + w_y^j,$$

where $w_x^j, w_y^j \sim \mathcal{U}([-0.001, 0.001])$.

- interaction-aware approach
- a platoon of 9 vehicles
- reachable over-approximations for $t \in [0, 1.5]$

N (units)	# of states	Our Approach (s)	POLAR (s)	JuliaReach (s)
4	16	1.369	14.182	12.579
9	36	3.144	43.428	59.929
20	80	9.737	316.337	–
50	200	46.426	4256.435	–

Table: Run-time comparison



POLAR = C. Huang et al., ATVA 2022

JuliaReach = C. Schilling et al., AAAI 2022

- reachability as a framework for safety certification of autonomous systems
- developed computationally efficient and scalable approaches for reachability using monotone system theory
- run-time verification of neural network controlled systems
- capture stabilizing effect of learning-based components