# On small-time local controllability

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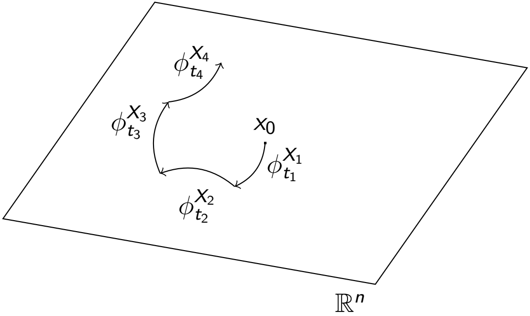
# Introduction

## Definition

A **control system** on $\mathbb{R}^n$ is a finite family of vector fields $\Sigma = \{X_1, X_2, \ldots, X_m\}$.

- We assume that the vector fields $X_1, X_2, \ldots, X_m$ are real analytic.
- A trajectory of $\Sigma$ is a concatenation of integral curves of the vector fields $\{X_1, X_2, \ldots, X_m\}$.



# Reachable sets

Given a control system $\Sigma = \{X_1, X_2, \ldots, X_m\}$ on $\mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$, we define

- Reachable set of $\Sigma$ from the point $x_0$:

$$\mathrm{R}_\Sigma(x_0) = \{\phi_{t_1}^{X_{i_1}} \circ \phi_{t_2}^{X_{i_2}} \circ \ldots \circ \phi_{t_k}^{X_{i_k}}(x_0) \mid t_i > 0,\ i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, m\}\}.$$

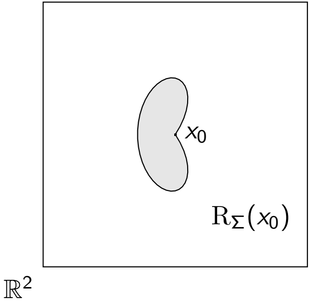
- Reachable set of $\Sigma$ in times less than $T$ from the point $x_0$:

$$\mathrm{R}_\Sigma(< T, x_0) = \{\phi_{t_1}^{X_{i_1}} \circ \phi_{t_2}^{X_{i_2}} \circ \ldots \circ \phi_{t_k}^{X_{i_k}}(x_0) \mid t_i > 0,\ \sum_{i=1}^{k} t_k < T,\ i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, m\}\}.$$

# Accessibility

## Local accessibility

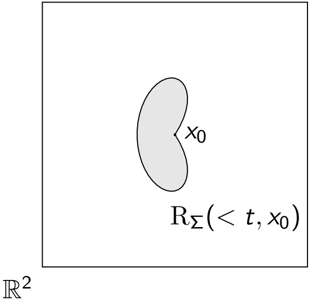
A control system $\Sigma$ is **locally accessible** from $x_0$ if $\mathrm{R}_\Sigma(x_0)$ has nonempty interior.



# Accessibility

## Small-time local accessibility

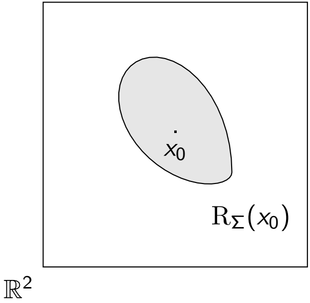
A control system $\Sigma$ is **small-time locally accessible** from $x_0$ if, for every small enough $t$, the set $\mathrm{R}_\Sigma(< t, x_0)$ has nonempty interior.



# Controllability

**Local controllability**

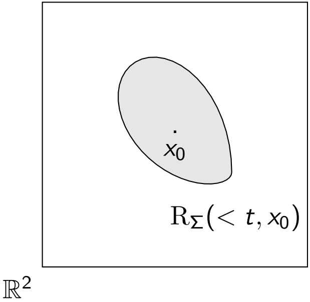
A control system $\Sigma$ is **locally controllable** from $x_0$ if $\mathrm{R}_\Sigma(x_0)$ contains a neighbourhood of $x_0$.



# Controllability

**Small-time local controllability**

A control system $\Sigma$ is **small-time locally controllable** from $x_0$ if, for small enough $t$, the set $\mathrm{R}_\Sigma(< t, x_0)$ contains a neighbourhood of $x_0$.



# Characterization of small-time local accessibility

- Small-time local accessibility of a real analytic control system $\Sigma$ from $x_0$ can be characterized in terms of Lie brackets of vector fields of the family $\Sigma$ at the point $x_0$.

**Theorem (H. J. Sussmann and V. Jurdjuvic 1972)**

A real analytic control system $\Sigma = \{X_1, X_2, \ldots, X_m\}$ is small-time locally accessible from $x_0$ if and only if

$$\operatorname{span}\left(\operatorname{Lie}(\{X_1, X_2, \ldots, X_m\})\right)(x_0) = \mathbb{R}^n.$$

- Is there a similar characterization for STLC of $\Sigma$ form $x_0$ using the Lie brackets of vector field of $\Sigma$ at $x_0$?

# Small-time local controllability

- Sufficient condition:
  1. H. J. Sussmann 1978, 1983, 1986,
  2. R. M. Bianchini and G. Stefani 1993,
  3. R. Hirschorn and A. D. Lewis 2004,
  4. M. I. Krastanov 2009.
- Necessary condition:
  1. G. Stefani 1986,
  2. M. Kawski 1987,
  3. M. I. Krastanov 1998.
- Necessary and Sufficient conditions for some specific classes of systems:
  1. C. O. Aguilar and A. D. Lewis 2012,

# Reachability and finite differentiation

- A nice feature of small-time local accessibility is that it is recognizable in finite number of differentiation.

**Example**

Consider the system $\Sigma = \{X_1, X_2\}$ on $\mathbb{R}^2$ such that

$$X_1(x,y) = \frac{\partial}{\partial x}, \quad X_2(x,y) = -x\frac{\partial}{\partial y},$$

It is easy to see that

$$[X_1, X_2](0,0) = -\frac{\partial}{\partial y},$$

and

$$\operatorname{span}\{X_1(0,0), [X_1, X_2](0,0)\} = \mathbb{R}^2.$$

Therefore $\Sigma$ is small-time locally accessible from $(0,0)$.

# Reachability and finite differentiation

**Example**

Now consider another control system $\Theta = \{Y_1, Y_2\}$ defined as

$$Y_1(x,y) = \frac{\partial}{\partial x}, \quad X_1(x,y) = \frac{\partial}{\partial x},$$
$$Y_2(x,y) = \left(x + x^2 + xy\right)\frac{\partial}{\partial y}, \quad X_2(x,y) = x\frac{\partial}{\partial y},$$

Then

$$[Y_1, Y_2](0,0) = -\frac{\partial}{\partial y},$$

and therefore $\Theta$ is small-time locally accessible from $(0,0)$.

- Conclusion: any perturbation of $\Sigma$ around $(0,0)$ by terms of order 2 or higher is still small-time locally accessible.

# Reachability and finite differentiation

- Do we have similar feature for STLC?

**Conjecture A (A. Agrachev 1999)**

Let $\Sigma$ be a real analytic control system which is STLC from $x_0$. Then there exists $N \in \mathbb{N}$ such that any other control system $\Theta$ with the same Taylor polynomials of order $N$ around $x_0$ is STLC from $x_0$?

# Control Variations

- A useful tool for studying STLC is **control variation**.

**Control Variations**

Let $\mathcal{U} = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$. Then a control variation is a map $u_s : [0, s] \to \mathcal{U}$.

- we define the time-varying vector field $X(t, x, u_s)$ as

$$X(t, x, u_s) = X^i(x), \qquad \text{if } u_s(t) = (0, 0, \ldots, 1, 0, \ldots, 0).$$

- Control variation: approximate the reachable sets of a system for small-enough time.

# Control Variations

## Higher-order tangent vectors

Let $u_s$ be a control variation for $\Sigma$ and $m \in \mathbb{Z}_{\geq 0}$. Let $x(t, u_s)$ be the solution of the initial value problem

$$\frac{dx}{dt}(t) = X(t, u_s).$$

Then $v \in \mathbb{R}^n$ is called an $m$-th order tangent vector to $\Sigma$ at point $x_0$ if we have

$$x(s, u_s) = x_0 + vs^m + o(s^m).$$

where $\lim_{s \to 0} \frac{o(s^m)}{s^m} = 0$.

- the cone generated by all $m$-th order tangent vectors of $\Sigma$ at point $x_0$ is denoted by $K^m_{\Sigma,x_0}$.
- For $l \leq m$, we have $K^m_{\Sigma,x_0} \subseteq K^l_{\Sigma,x_0}$.

# Open mapping theorem

- Control variations can be used to find admissible **directions** in the reachable set of the system for small enough time.
- How to show STLC using control variations? We use a suitable open mapping theorem.

**Theorem (M. Kawski 1990)**

If $K^m_{\Sigma,x_0} = \mathbb{R}^n$, then there exists $C, T > 0$ such that

$$\mathrm{B}(x_0, Ct^m) \subseteq \mathrm{R}_\Sigma(< t, x_0), \qquad \forall t \in [0, T].$$

# Example

## Example

Consider the control system $\Sigma = \{X_1, X_2, X_3, X_4\}$ on $\mathbb{R}^2$.

$$X_1(x,y) = \tfrac{\partial}{\partial x}, \quad X_3(x,y) = -\tfrac{\partial}{\partial x},$$
$$X_2(x,y) = x\tfrac{\partial}{\partial y}, \quad X_4(x,y) = -x\tfrac{\partial}{\partial y}.$$

Then we have

$$\mathrm{cone}\,(X_1(0,0), X_2(0,0), X_3(0,0), X_4(0,0)) = \text{x-axis}.$$

By choosing the control variation $u_s : [0,s] \to \mathcal{U}$ as

$$u_s(t) = \begin{cases} (1,0,0,0) & t \in [0, \frac{s}{4}), \\ (0,1,0,0) & t \in [\frac{s}{4}, \frac{s}{2}), \\ (0,0,1,0) & t \in [\frac{s}{2}, \frac{3s}{4}), \\ (0,0,0,1) & t \in [\frac{3s}{4}, s). \end{cases}$$

# Example

## Example

Then we have

$$x(s,u_s) = \phi^{X_1}_{\frac{s}{4}} \circ \phi^{X_2}_{\frac{s}{4}} \circ \phi^{-X_1}_{\frac{s}{4}} \circ \phi^{-X_2}_{\frac{s}{4}}$$
$$= \frac{1}{16}[X_2, X_1](0,0)s^2 + o(s^2) = \frac{1}{16}\frac{\partial}{\partial y}s^2 + o(s^2).$$

Thus $[X_2, X_1](0,0)$ is a tangent vector of order 2. Similarly, we can show that $[X_1, X_2](0,0)$ is a tangent vector of order 2.

$$\mathrm{cone}(X_1(0,0), X_3(0,0), [X_1, X_2](0,0), [X_2, X_1](0,0)) = \mathbb{R}^2.$$

This implies that $\Sigma$ is STLC form $(0,0)$. Moreover, for small enough $t$, we have

$$\mathrm{B}(\mathbf{0}, Ct^2) \subseteq \mathrm{R}_\Sigma(< t, \mathbf{0})$$

# The growth rate of reachable sets

Suppose that we have a real analytic control system $\Sigma$. We find a finite family of control variations for $\Sigma$ at point $x_0$ such that

1. their associated higher-order tangent vectors are of order at most $m$,
2. the cone generated by these higher-order tangent vectors is the whole $\mathbb{R}^n$.

Then $\Sigma$ is STLC form $x_0$. Moreover, there exists $C, T > 0$ such that

$$\mathrm{B}(x_0, Ct^m) \subseteq \mathrm{R}_\Sigma(< t, x_0), \qquad \forall t \in [0, T].$$

# The growth rate of reachable sets

- Can we prove STLC of every system using the above method?

**Conjecture B (A. Agrachev 1999)**

Let $\Sigma$ be a real analytic control system which is STLC from $x_0$. Then there exist $N \in \mathbb{Z}_{>0}$ and $C, T > 0$ such that

$$\mathrm{B}(x_0, Ct^N) \subseteq \mathrm{R}_\Sigma(< t, x_0), \qquad \forall t \in [0, T].$$

# Main theorem

**Theorem**

Let $\Sigma = \{X_1, X_2, \ldots, X_m\}$ be a real analytic control system and there exist $C, T > 0$ such that

$$\mathrm{B}(x_0, Ct^N) \subseteq \mathrm{R}_\Sigma(< t, x_0), \qquad \forall t \in [0, T].$$

Let $\Theta = \{Y_1, Y_2, \ldots, Y_m\}$ be another real analytic control system such that

- for every $i \in \{1, 2, \ldots, m\}$, the first $N$-terms in the Taylor series of $X_i$ and $Y_i$ around $x_0$ agree.

Then $\Theta$ is STLC from $x_0$.

- In particular, this theorem proves that conjecture B implies conjecture A.

# Brouwer fixed-point theorem

- The Brouwer fixed-point theorem is one of the most fundamental existence theorem in mathematics.

**Brouwer fixed-point theorem**

Let $K$ be a compact and convex set in $\mathbb{R}^n$ and $f : K \to K$ is continuous. Then $f$ has at least one fixed point (i.e., there exists $x \in K$ such that $f(x) = x$).

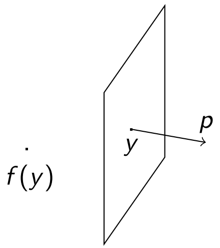
- There are many different generalizations of the Brouwer fixed-point theorem.

**Definition**

A map $f : K \to K$ is **half-continuous** if, for every $x \in K$ such that $f(x) \neq x$, there exist $p \in \mathbb{R}^n$ and a neighbourhood $U$ of $x$ such that

$$p.\,(f(y) - y) > 0, \qquad \forall y \in U.$$

# Brouwer fixed-point theorem



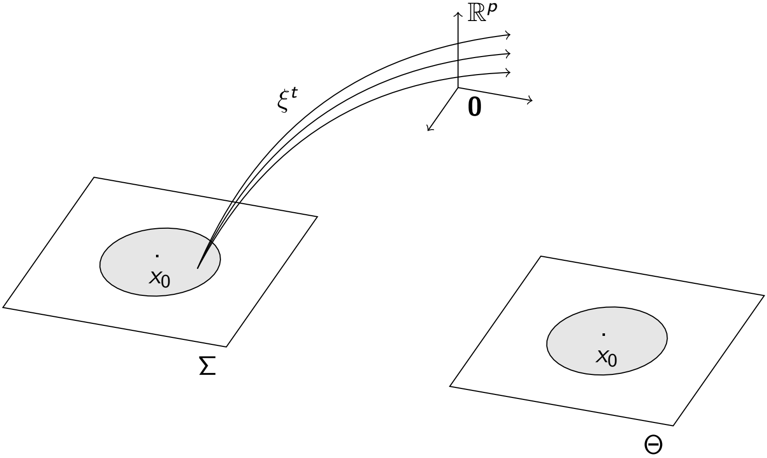

**Brouwer fixed-point theorem (P. Bich 2006)**

Let $K$ be a compact and convex set in $\mathbb{R}^n$ and $f : K \to K$ be half-continuous. Then $f$ has at least one fixed point.

Idea of the proof: For every $t$ small enough, we have
$\mathrm{B}(x_0, \frac{C}{2}t^N) \subseteq \mathrm{R}_\Theta(< t, x_0)$

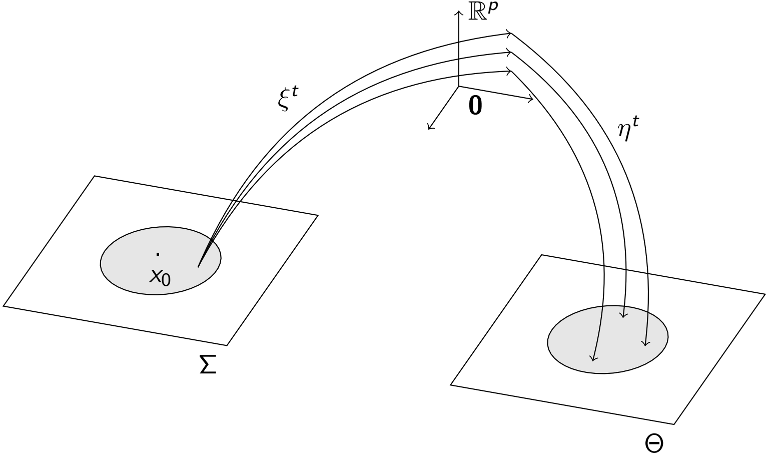
# Sketch of proof

- Fix $t > 0$ small enough.



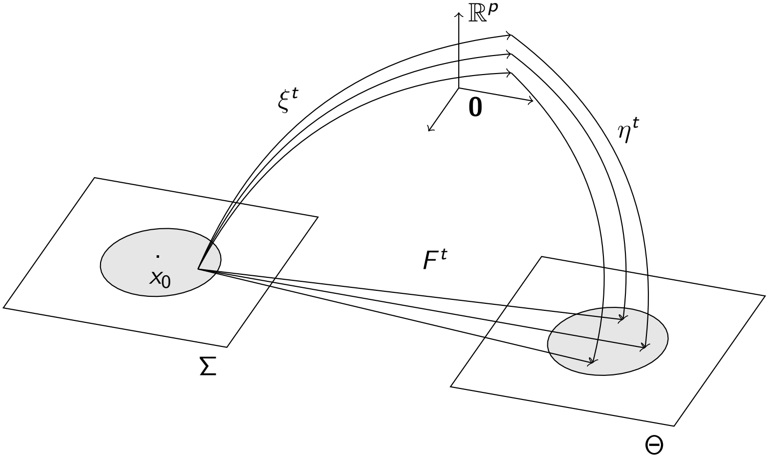

- $\xi^t$ maps every $\phi^{X_{i_1}}_{t_1} \circ \phi^{X_{i_2}}_{t_2} \circ \dots \phi^{X_{i_k}}_{t_k}(x_0)$ to $(t_1, t_2, \dots, t_k, 0, 0, \dots, 0)$ in $\mathbb{R}^p$.

# Sketch of proof




- $\eta^t$ maps every $(t_1, t_2, \ldots, t_p)$ in $\mathbb{R}^p$ to $\phi_{t_1}^{Y_{i_1}} \circ \phi_{t_2}^{Y_{i_2}} \circ \ldots \phi_{t_p}^{Y_{i_p}}(x_0)$.

# Sketch of proof




- $F^t : \mathrm{B}(x_0, Ct^N) \rightrightarrows \mathrm{R}_\Theta(< t, x_0)$ defined as $F^t = \eta^t \circ \xi^t$.

# Sketch of proof

- Fix $y \in \mathrm{B}(x_0, \frac{C}{2}t^N)$ and define $G_y^t : \mathrm{B}(x_0, Ct^N) \rightrightarrows \mathbb{R}^p$ as

$$G_y^t(x) = x - F^t(x) + y.$$

- $G_y^t$ is multi-valued and has a half-continuous selection $g_y^t : \mathrm{B}(x_0, Ct^N) \to \mathbb{R}^p$.
- $\Sigma$ and $\Theta$ have the same Taylor polynomials of order $N$ around $x_0$, therefore $g_y^t\left(\mathrm{B}(x_0, Ct^N)\right) \subseteq \mathrm{B}(x_0, Ct^N)$.
- By the generalized Brouwer fixed point theorem $g_y^t$ has a fixed point.

$$x \in x - F^t(x) + y \quad \Rightarrow \quad y \in F^t(x)$$

- $y \in \mathrm{R}_\Theta(< t, x_0)$.