Network Critical Slowing Down:

Data-driven Detection of Critical Transitions in Networks

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Workshop on Robustness, Resilience, and Early Warning Signals in Natural Dynamical Networks

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Acknowledgment

Mohammad Pirani
University of Ottawa

Mohammad Pirani and SJ. Network Critical Slowing Down: Data-Driven Detection of Critical Transitions in Nonlinear Networks. TCNS, 2024.

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Small condition change $\implies$ a large shift in state of system

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- spontaneous systemic failure such as epileptic seizures (Litt, B. et al., 2001 and McSharry, P. E. et al. 2003)
- crashes in financial markets (Kambhu, J. et al., 2007 and May, R. M. et al., 2008)
- abrupt shifts in ocean circulations (Lenton, T. M. et al., 2008)
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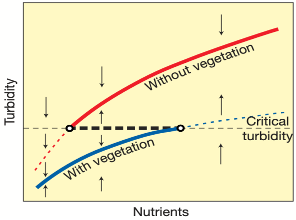
Lake Zwemlust Experiment:



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In this talk: Dynamical system perspective toward critical transitions.

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Critical transitions

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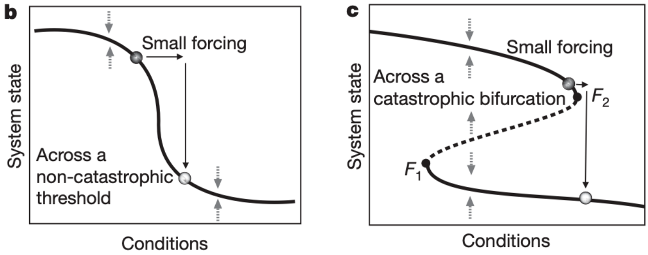

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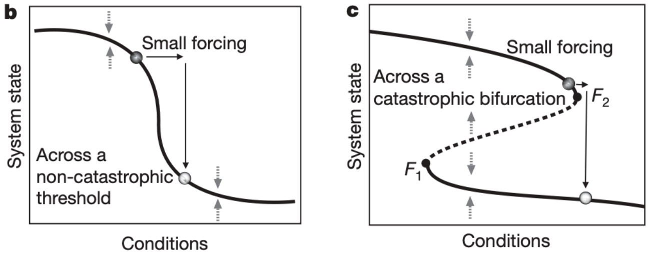

In this talk: we focus on **bifurcations** (cases (c))

Question:

How can we detect these transitions caused by bifurcations?

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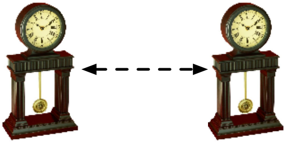
- [Early work] Wissel, C. A universal law of the characteristic return time near thresholds. Oecologia 65, 101–107 (1984).
- [In ecology] Carpenter, S. R. & Brock, W. A. Rising variance: a leading indicator of ecological transition. Ecol. Lett. 9, 308–315 (2006).
- [In climat change] Dakos, V. et al. Slowing down as an early warning signal for abrupt climate change. Proc. Natl Acad. Sci. USA 105, 14308–14312 (2008).
- [more than 6000 citations!] Scheffer M. et al., Early-warning signals for critical transitions, Nat. Rev., Vol 461, (2009)

Pendulum clocks: "an odd kind of sympathy"

[Christiaan Huygens, Horologium Oscillatorium, 1673]



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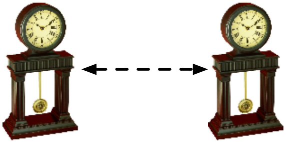

Kuramoto model

1. two **oscillators** with phases θ_1 and θ_2,
2. with natural frequencies $\omega_1(\eta)$ and $\omega_2(\eta)$,
3. **coupling** with strength K:

$$\dot{\theta}_1 = \omega_1(\eta) - K\sin(\theta_1 - \theta_2),$$
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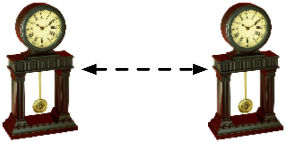

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Reduced-order: $\dot{\phi} = \overline{\omega}(\eta) - 2K\sin(\phi)$ $\quad\phi := \theta_1 - \theta_2$ and $\overline{\omega}(\eta) := \omega_1(\eta) - \omega_2(\eta)$

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Equilibrium analysis

- $\frac{\overline{\omega}(\eta)}{2K} < 1$: one stable and one unstable equilibrium point
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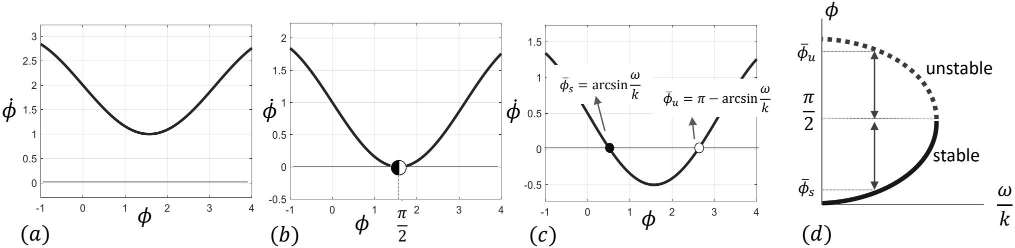

Change in the condition η leads to a **saddle-node bifurcation**.

Key detection idea: Dramatic increase in rate of recovery form small perturbations

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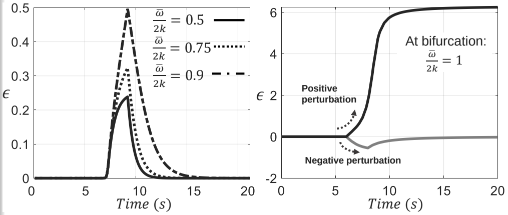
Perturbation Dynamics: $\dot{\epsilon} \approx D_x f(x^*(\eta))\ \epsilon$

Perturbation for coupled oscillators:

$$\dot{\epsilon} \approx -2K \cos(\arcsin(\tfrac{\overline{\omega}(\eta)}{2K}))\epsilon$$

- when $\frac{\overline{\omega}(\eta)}{2K} < 1$, $\epsilon = 0$ is exponentially stable with rate $2K \cos(\arcsin(\frac{\overline{\omega}(\eta)}{2K}))$.
- As $\frac{\overline{\omega}(\eta)}{2K} \to 1$ we have $2K \cos(\arcsin(\frac{\overline{\omega}(\eta)}{2K})) \to 0$.



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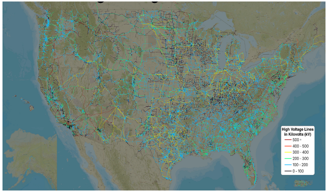
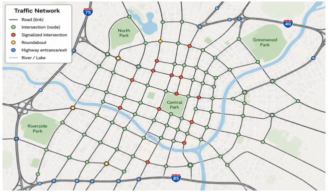
In many complex systems, although the model may be inaccurate, we **know** the underlying network structure.

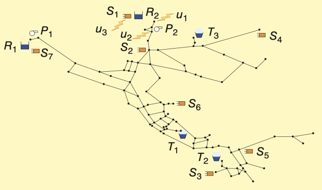
Power grids



Traffic networks



Water supply network



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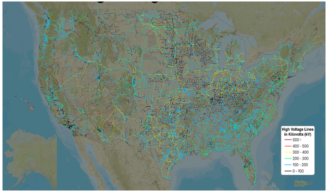

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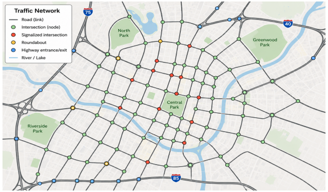

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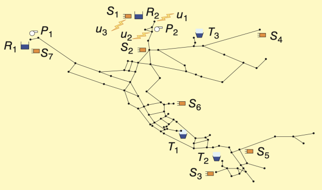

Question: exploit the network structure to detect and localize critical transitions?

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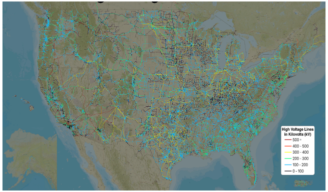

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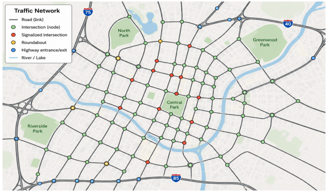

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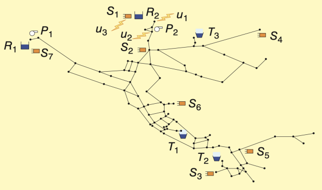

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2. **physics:**
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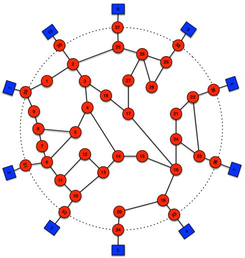
3. simplifying assumptions:
   1. lossless and inductive lines with admittances Y_{ij}
   2. decoupling of phase and voltage dynamics



New England IEEE 39-bus

Structure-Preserving Model [A. Bergen & D. Hill, 1981]:

Active power dynamics

Inverters: $$\Lambda_i \dot{\theta}_i = p_i(\eta) - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

Loads: $$\tau_i \dot{\theta}_i = p_i(\eta) - \sum_j a_{ij} \sin(\theta_i - \theta_j)$$

where

Active power capacity of line (i, j): $a_{ij} = |Y_{ij}| V_i V_j$

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p(\eta))_{(ij)} = p(\eta)_i - p(\eta)_j$

Edge weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian matrix: $L = B\mathcal{A}B^\top$

Time-scale matrix: $D = \mathrm{diag}(\{\Lambda_i\}, \{\tau_i\})$

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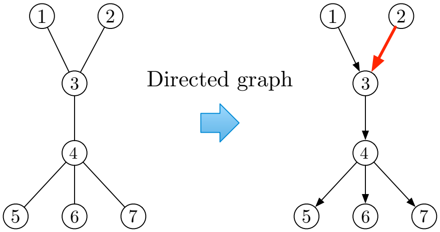

nodes

$$B^\top = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \to 3 \\ 2 \to 3 \\ 3 \to 4 \\ 4 \to 5 \\ 4 \to 6 \\ 4 \to 7 \end{array}$$

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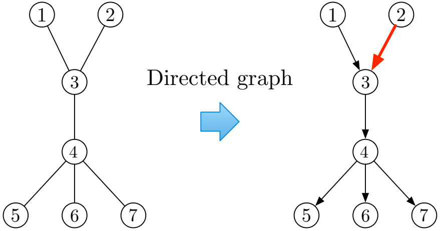

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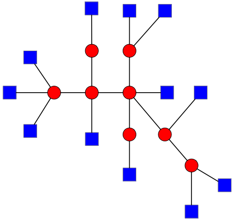
Active power dynamics: $\dot{\theta} = D^{-1}\left(p(\eta) - B\mathcal{A}\sin(B^\top\theta)\right)$

In this talk: power grids with acyclic topology

Example: distribution grid

Equilibrium points: $p(\eta) = B\mathcal{A}\sin(B^\top\theta)$



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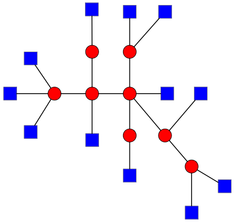

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Multiply both side into $B^\top L^\dagger$

$$B^\top L^\dagger p(\eta) = B^\top L^\dagger B\mathcal{A}\sin(B^\top\theta)$$

$B^\top L^\dagger B\mathcal{A}$ is a projection matrix

For acyclic topology: $B^\top L^\dagger B\mathcal{A} = I_n$

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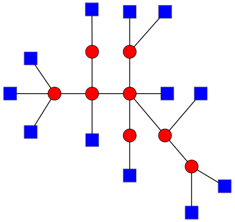

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$$\text{if } \|B^\top L^\dagger p(\eta)\|_\infty \leq 1 \quad \Longrightarrow \quad B^\top\theta^*(\eta) = \arcsin\left(B^\top L^\dagger p(\eta)\right)$$

$$\text{if } \|B^\top L^\dagger p(\eta)\|_\infty \leq 1 \quad \Longrightarrow \quad \theta_i^*(\eta) - \theta_j^*(\eta) = \arcsin\left(B^\top L^\dagger p(\eta)\right)_e$$

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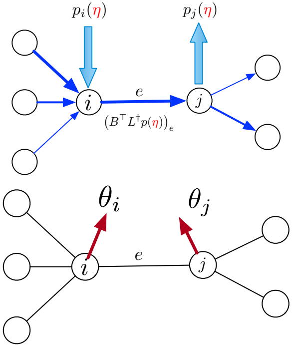
$\left(B^\top L^\dagger p(\eta)\right)_e$ = active power flow from bus i to bus j passes through the line $e = i \to j$.

$\left(B^\top \theta\right)_e = \theta_i^*(\eta) - \theta_j^*(\eta)$ = voltage angle difference between bus i and bus j



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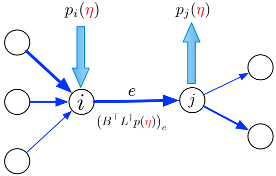

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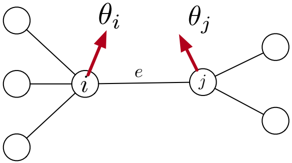

Frequency synchronization:

$\sin$(voltage angle differences between i and j) = active power flows on e

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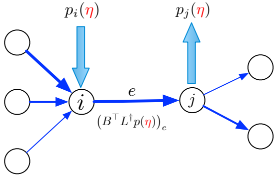

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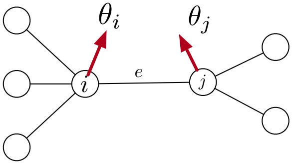

Frequency synchronization:

$\sin$(voltage angle differences between i and j) = active power flows on e

if active power flow on $e : i \to j \quad > 1 \quad \Longrightarrow \quad$ the line $e : i \to j$ **overheats**

Assumption: generic violation

- $\eta \leq \eta^*$: all active power flows $B^\top L^\dagger p(\eta) \leq 1$
- $\eta > \eta^*$: only one line e that $\left(B^\top L^\dagger p(\eta)\right)_e > 1$

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Saddle-node bifurcation

- $\eta < \eta^*$: power grid has one stable $\theta^*(\eta)$ and one unstable eq point $\theta^{**}(\eta)$:
  1. $B^\top \theta^*(\eta) = \arcsin(B^\top L^\dagger p(\eta))$
  2. $[B^\top \theta^{**}(\eta)]_\ell = \begin{cases} \arcsin(B^\top L^\dagger p(\eta))_\ell & \ell \neq e \\ \pi - \arcsin(B^\top L^\dagger p(\eta))_\ell & \ell = e \end{cases}$
- $\eta = \eta^*$: power grid has one unstable eq point $\theta^*(\eta)$
- $\eta > \eta^*$: power grid has no eq point

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Saddle-node bifurcation

- $\eta < \eta^*$: power grid has one stable $\theta^*(\eta)$ and one unstable eq point $\theta^{**}(\eta)$:
  1. $B^\top \theta^*(\eta) = \arcsin(B^\top L^\dagger p(\eta))$
  2. $[B^\top \theta^{**}(\eta)]_\ell = \begin{cases} \arcsin(B^\top L^\dagger p(\eta))_\ell & \ell \neq e \\ \pi - \arcsin(B^\top L^\dagger p(\eta))_\ell & \ell = e \end{cases}$
- $\eta = \eta^*$: power grid has one unstable eq point $\theta^*(\eta)$
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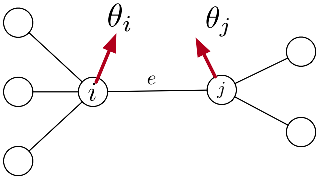

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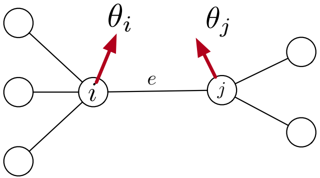

- $\eta < \eta^*$: line e is safe
- $\eta = \eta^*$: max capacity on e
- $\eta > \eta^*$: line e overheats

Original Dynamics: $\dot{\theta} = D^{-1}\left(p(\eta) - B\mathcal{A}\sin(B^{\top}\theta)\right)$

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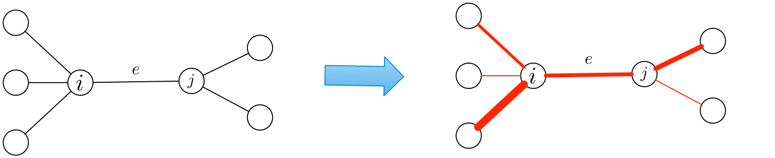
weight of line $e = V_iV_j|Y_{ij}|\cos(\theta_i^*(\eta) - \theta_j^*(\eta))$



$L = B^\top \mathcal{A} B$

$L(\eta) = B^\top \mathcal{A}\,\mathrm{diag}(B^\top\theta^*(\eta))B$

$L(\eta)$ is a positive semi-definite matrix

eigenvalues of $L(\eta)$: $\lambda_1(\eta) \leq \lambda_2(\eta) \leq \ldots \leq \lambda_n(\eta)$

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Theorem

For the perturbation of a power grid with acyclic topology as $t \to \infty$:

$$\epsilon(t) \approx \frac{1}{n} \left(\mathbb{1}_n^\top \epsilon(0)\right) \mathbb{1}_n + e^{-\lambda_2(\eta)t} \left(v_2^\top(\eta)\epsilon(0)\right) v_2(\eta)$$

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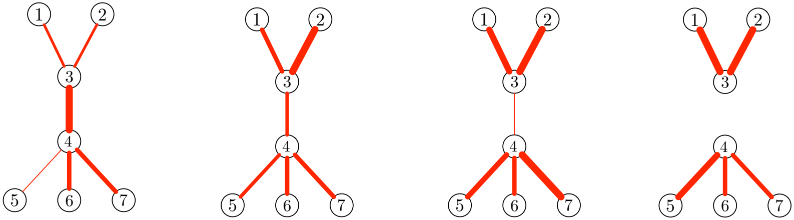


$\eta \ll \eta^*$ $\eta < \eta^*$ $\eta \approx \eta^*$ $\eta = \eta^*$

$$\lim_{\eta \to \eta^*} \lambda_2(\eta) = 0$$

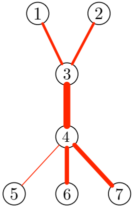
$\epsilon(t) \approx \frac{1}{n}\left(\mathbb{1}_n^\top \epsilon(0)\right)\mathbb{1}_n + e^{-\lambda_2(\eta)t}\left(v_2^\top(\eta)\epsilon(0)\right)v_2(\eta)$: how does $v_2(\eta)$ changes as $\eta \to \eta^*$?

Renewable Power Grids

Effect of bifurcation on Fiedler vector

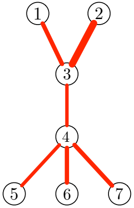
$\epsilon(t) \approx \frac{1}{n}\left(\mathbb{1}_n^\top \epsilon(0)\right)\mathbb{1}_n + e^{-\lambda_2(\eta)t}\left(v_2^\top(\eta)\epsilon(0)\right)v_2(\eta)$: how does $v_2(\eta)$ changes as $\eta \to \eta^*$?

$$\begin{bmatrix} 0.2493 \\ 0.2493 \\ 0.1597 \\ 0.0710 \\ -0.9104 \\ 0.0905 \\ 0.0905 \end{bmatrix}$$



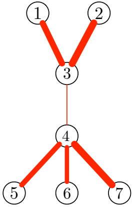
$\eta \ll \eta^*$

$$\begin{bmatrix} 0.5182 \\ 0.4320 \\ 0.3383 \\ -0.2210 \\ -0.3904 \\ -0.3385 \\ -0.3385 \end{bmatrix}$$



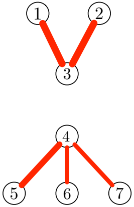
$\eta < \eta^*$

$$\begin{bmatrix} 0.4494 \\ 0.4447 \\ 0.4142 \\ -0.3044 \\ -0.3351 \\ -0.3420 \\ -0.3268 \end{bmatrix}$$



$\eta \approx \eta^*$

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-1} \\ \mathbf{-1} \\ \mathbf{-1} \\ \mathbf{-1} \end{bmatrix}$$



$\eta = \eta^*$

$$\lim_{\eta \to \eta^*} v_2(\eta) = \frac{1}{\sqrt{n}}\chi^e \text{ where } \chi_k^e = \begin{cases} +1 & i \mapsto k \\ -1 & j \mapsto k \end{cases}$$

punchline: close to η^*, $\operatorname{sgn}(v_2(\eta))$ localizes the line e.

$$\epsilon(t) \approx \frac{1}{n}\left(\mathbb{1}_n^\top \epsilon(0)\right)\mathbb{1}_n + e^{-\lambda_2(\eta)t}\left(v_2^\top(\eta)\epsilon(0)\right)v_2(\eta).$$

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- $\eta \ll \eta^*$: As $t \to \infty$ we have $\epsilon(t) \approx \frac{1}{n}\left(\mathbb{1}_n^\top \epsilon(0)\right)\mathbb{1}_n$.
- η close to η^*: As $t \to \infty$ we have $\epsilon(t) \approx \frac{1}{n}\left(\mathbb{1}_n^\top \epsilon(0)\right)\mathbb{1}_n + e^{-\lambda_2(\eta)t}\left(v_2^\top(\eta)\epsilon(0)\right)v_2(\eta)$.
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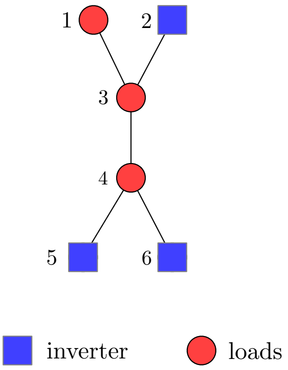
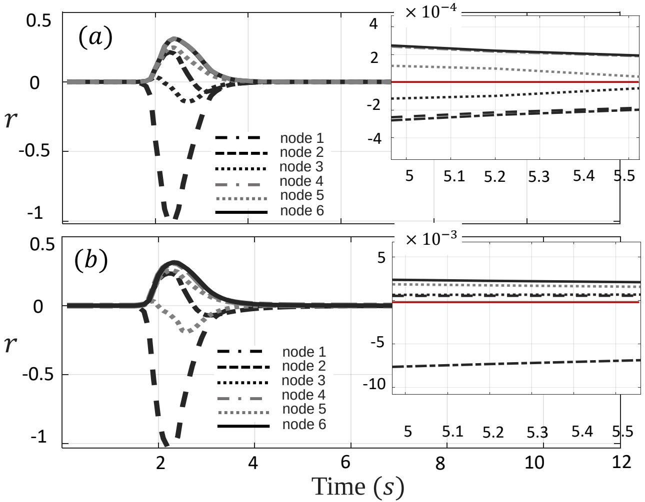
Key for detection and localization: when η close to η^* then $\lambda_2(\eta) \approx 0$ and $v_2(\eta) \approx \chi^e$:

As $t \to \infty$ we have $r(t) := \epsilon(t) - \frac{1}{n}\left(\mathbb{1}_n^\top \epsilon(0)\right)\mathbb{1}_n \approx \frac{1}{n}\left((\chi^e)^\top \epsilon(0)\right)\chi^e$

As $t \to \infty$, $\boxed{\mathrm{sgn}(r(t)) \approx \chi^e}$





As $t \to \infty$ we have
$\mathrm{sgn}(r(t)) = [+1 -1 +1 +1 +1 +1]^\top$

Bifurcation in the line $3 \to 2$

Perturbation $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ has applied from $t = 2\mathrm{s}$ to $t = 3\mathrm{s}$. **Above**: far from the bifurcation. **Below:** close to the bifurcation

Conclusions

- Notion of **Network Critical Slowing Down** as a generalization of classical critical slowing down for networked dynamical systems.
- Graph-theoretic tools to **detect** and **localize** critical transitions.
- A **data-driven** framework for detecting and localizing critical transitions in power networks.

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Future Directions:

- Generalization to broader classes of Laplacian-based nonlinear systems with directed interconnections.
- Extension to stochastic network dynamics, leveraging covariance matrices and higher-order moments for the detection of critical transitions.
- Integration with feedback control for resilience enhancement and mitigation of critical transitions.

Thank you for your attention!

Back up Slides

Key detection idea: Dramatic increase in fluctuation autocorrelation for stochastic noise

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$\mathbb{E}(e_t) = 0$ and $\lim_{t\to\infty} \mathrm{Var}(e_t) = \frac{\sigma^2}{1-\alpha(\eta)^2}$

- far from bifurcation: $\alpha(\eta) \ll 1$
- close to bifurcation: $\alpha(\eta) \to 1$ and very large $\mathrm{Var}(e_t)$