Reachability Analysis of Neural Network Controlled Systems: A Mixed Monotone Contracting Approach

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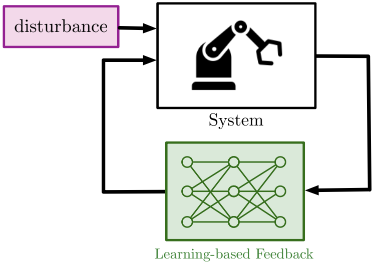
SJ and A. Harapanahalli and S. Coogan. Interval Reachability of Nonlinear Dynamical Systems with Neural Network Controllers. L4DC, 2023.

SJ and A. Davydov and F. Bullo. Non-Euclidean Contraction Theory for Monotone and Positive Systems. IEEE Transactions on Automatic Control, Dec. 2022

SJ and S. Coogan. Monotonicity and contraction on polyhedral cones. arXiv, 2021.

- Neural Networks as controllers in safety-critical applications (self driving vehicle and mobile robots)

Goal: ensure and verify *safety* of the closed-loop system



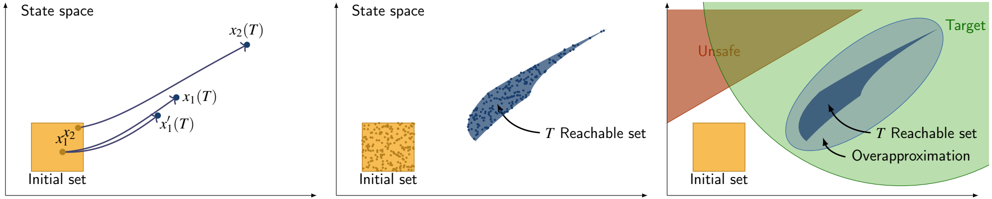
Features of neural network controllers:

- large # of parameters with nonlinearity
- sensitive wrt to input perturbations
- limited closed-loop safety guarantees

Challenges

Sound and computationally efficient methods for safety verification

System : $\dot{x} = f(x, w)$ **State** : $x \in \mathbb{R}^n$ **Disturbance** : $w \in \mathcal{W} \subseteq \mathbb{R}^m$



- reachable sets characterize evolution of the system

$$\mathcal{R}^f(t, \mathcal{X}_0) = \{x_w(t) \mid x_w(\cdot) \text{ is a traj of the system for some } w \text{ with } x_0 \in \mathcal{X}_0\}$$

- over-approximation of reachable sets for safety and verification
- reachability of dynamical system is an old problem with several classical approaches

not scalable to large-scale nonlinear systems

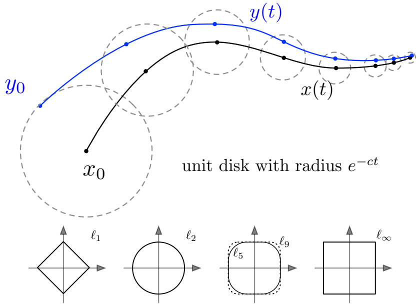
Contraction Theory

A framework for stability analysis

$\dot{x} = f(x, w)$ is contracting wrt $\|\cdot\|$ with rate c if
the dist between every two traj is decreasing/increasing with exp rate c wrt $\|\cdot\|$

Common applications

- convergence to reference trajectories
- efficient equilibrium point computation
- input-output robustness
- entrainment to periodic orbits



In this talk: contraction theory for reachability analysis

How to characterize contractivity using vector fields?

Matrix measure

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a norm $\|\cdot\|$:

$$\mu_{\|\cdot\|}(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

Given $\eta \in \mathbb{R}^n_{\geq 0}$

$$\mu_{2,\eta}(A) = \tfrac{1}{2}\lambda_{\max}(\mathrm{diag}(\eta)A + A^\top \mathrm{diag}(\eta))$$

$$\mu_{1,\eta}(A) = \max_j \left(a_{jj} + \sum\nolimits_{i\neq j} |a_{ij}|\tfrac{\eta_j}{\eta_i}\right)$$

$$\mu_{\infty,\eta}(A) = \max_i \left(a_{ii} + \sum\nolimits_{j\neq i} |a_{ij}|\tfrac{\eta_j}{\eta_i}\right)$$

- directional derivative of matrix norm $\|\cdot\|$ in direction of A at point I_n,
- **In the literature**: one-sided Lipschitz constant, logarithmic norm

Classical result

$\dot{x} = f(x,w)$ is contracting wrt $\|\cdot\|$ with rate c iff

$$\mu_{\|\cdot\|}\left(\tfrac{\partial f}{\partial x}(x,w)\right) \leq c, \qquad \text{for all } x, w$$

Contraction-based Reachability Analysis

A global bound

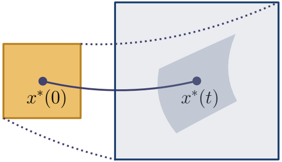
Assume $\mu_{\|\cdot\|}\left(\frac{\partial f}{\partial x}(x,w)\right) \leq c$ and $\left\|\frac{\partial f}{\partial w}(x,w)\right\| \leq \ell$

Theorem

If $\mathcal{X}_0 = B_{\|\cdot\|}(r_1, x_0^*)$ and $\mathcal{W} = B_{\|\cdot\|}(r_2, w^*)$, then

$$\mathcal{R}^f(t,\mathcal{X}_0) \subseteq B_{\|\cdot\|}(e^{ct}r_1 + \tfrac{\ell}{c}(e^{ct}-1)r_2, x^*(t))$$

where $x^*(\cdot)$ is the solution of $\dot{x} = f(x,w^*)$ with $x(0) = x_0^*$.



Proof: let $x(\cdot)$ be a traj of $\dot{x} = f(x,w)$. Using Taylor expansion, for $h \geq 0$

$$x(t+h) - x^*(t+h) = x(t) - x^*(t) + h\overbrace{\left(\int_0^1 D_x f(\tau x + (1-\tau)x^*)d\tau\right)}^{A(x,w)}(x(t)-x^*(t))$$

$$+ h\overbrace{\left(\int_0^1 D_w f(x, \tau w + (1-\tau)w^*)d\tau\right)}^{B(x,w)}(w-w^*) + \mathcal{O}(h^2)$$

Contraction-based Reachability

Proof continued

$$D^+\|x(t) - x^*(t)\| = \limsup_{h\to 0^+} \frac{\|x(t+h) - x^*(t+h)\| - \|x(t) - x^*(t)\|}{h}$$

$$= \limsup_{h\to 0^+} \frac{\|\left(I_n + hA(x,w)\right)\left(x(t) - x^*(t)\right) + hB(x,w)(w - w^*)\| - \|x(t) - x^*(t)\|}{h}$$

$$\leq \limsup_{h\to 0^+} \frac{\|\left(I_n + hA(x,w)\right)\left(x(t) - x^*(t)\right)\| + h\|B(x,w)\|\|w - w^*\| - \|x(t) - x^*(t)\|}{h}$$

$$\leq \mu_{\|\cdot\|}(A(x,w))\|x(t) - x^*(t)\| + \|B(x,w)\|\|w - w^*\|$$

$$\leq c\|x(t) - x^*(t)\| + \ell\|w - w^*\|$$

- generalized version of Grönwall's lemma
- overly conservative since c and ℓ are defined globally

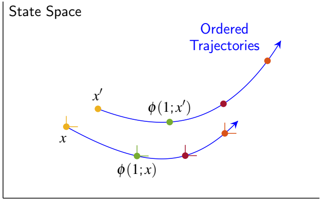
A dynamical system $\dot{x} = f(x, w)$ is monotone[1] if

$$x_u(0) \preceq_K y_w(0) \quad \text{and} \quad u \preceq_C w \quad \implies \quad x_u(t) \preceq_K y_w(t) \quad \text{for all time}$$

where $\preceq_K$ is the partial order with induced by the cone K.

A **polyhedral cone** has the form

$$K = \underbrace{\{y \in \mathbb{R}^n \mid H_K y \geq \mathbb{0}_p\}}_{\text{halfspace rep}} = \underbrace{\{V_K y \mid y \geq \mathbb{0}_p\}}_{\text{vertex rep}}$$



Monotonicity test

1. $H_K(\frac{\partial f}{\partial x}(x, w) + \alpha(x, w) I_n) V_K \geq \mathbb{0}_p$ for some $\alpha(x, w)$
2. $H_C \frac{\partial f}{\partial w}(x, w) V_C \geq \mathbb{0}_q$

[1] D. Angeli and E. Sontag, "Monotone control systems", TAC, 2003

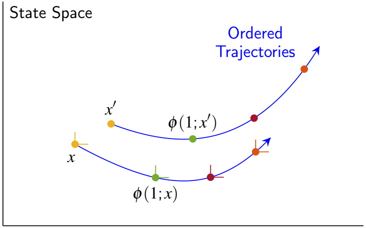
A dynamical system $\dot{x} = f(x, w)$ is monotone[1] if

$$x_u(0) \preceq_K y_w(0) \quad \text{and} \quad u \preceq_C w \quad \Longrightarrow \quad x_u(t) \preceq_K y_w(t) \quad \text{for all time}$$

where $\preceq_K$ is the partial order with induced by the cone K.

Monotonicity test (for standard partial order $\leq$)

1. $\frac{\partial f}{\partial x}(x, w)$ is Metzler (off-diag ≥ 0)
2. $\frac{\partial f}{\partial w}(x, w) \geq \mathbb{0}_m$



[1] D. Angeli and E. Sontag, "Monotone control systems", TAC, 2003

Theorem

For a monotone system with $\mathcal{W} = [\underline{w}, \overline{w}]$

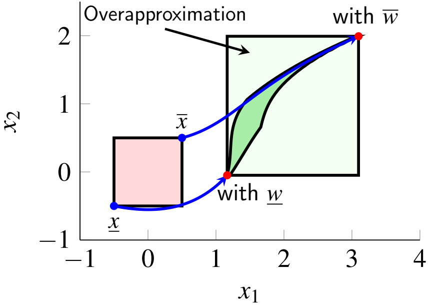
$$\mathcal{R}^f(t, [\underline{x}_0, \overline{x}_0]) \subseteq [x_{\underline{w}}(t), x_{\overline{w}}(t)]$$

where $x_{\underline{w}}(\cdot)$ (resp. $x_{\overline{w}}(\cdot)$) is the trajectory with disturbance $\underline{w}(\cdot)$ (resp. $\overline{w}(\cdot)$) starting at $\underline{x}_0$ (resp. $\overline{x}_0$)

Example:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_1 + w \\ x_1 \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$



Theorem

Let $\mathcal{X} = [\underline{x}_0, \overline{x}_0]$, $\mathcal{W} = [\underline{w}, \overline{w}]$, and $\eta \in \mathbb{R}^n_{\geq 0}$. Suppose that the monotone system $\dot{x} = f(x, w)$ is contracting with rate c wrt to $\|\cdot\|_{\infty,\eta}$, then

$$[x_{\underline{w}}(t), x_{\overline{w}}(t)] \subseteq B_{\|\cdot\|_{\infty,\eta}}(e^{ct} r_1 + \tfrac{\ell(e^{ct}-1)}{c} r_2, x^*(t))$$

where $x^*(\cdot)$ is the solution to $\dot{x} = f(x, w^*)$ with $x^*(0) = x_0^* \in \mathcal{X}_0$ and

$$r_1 = \max\{\|\underline{x} - x_0^*\|_{\infty,\eta}, \|\overline{x} - x_0^*\|_{\infty,\eta}\},$$
$$r_2 = \max\{\|\underline{w} - w^*\|_{\infty,\eta}, \|\overline{w} - w^*\|_{\infty,\eta}\}$$

Proof: both $x_{\underline{w}}(\cdot)$ and $x_{\overline{w}}(\cdot)$ are trajectories of the system

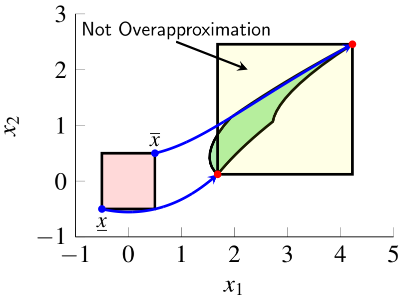
monotone reachability is at least as accurate as contraction reachability wrt diagonally weighted ℓ_∞-norms

- For non-monotone dynamical systems the extreme trajectories do not provide any over-approximation of reachable sets

Example:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2^3 - x_2 + w \\ x_1 \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$



- **Key idea:** embed the dynamical system on $\mathbb{R}^n$ into a dynamical system on $\mathbb{R}^{2n}$
- Assume $\mathcal{W} = [\underline{w}, \overline{w}]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$

Original system

$$\dot{x} = f(x, w)$$

Embedding system

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}),$$
$$\dot{\overline{x}} = \overline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$$

$\underline{d}, \overline{d}$ are **decomposition functions** s.t.

1. $f(x, w) = \underline{d}(x, x, w, w)$ for every x, w
2. cooperative: $(\underline{x}, \underline{w}) \mapsto \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$
3. competitive: $(\overline{x}, \overline{w}) \mapsto \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w})$
4. the same properties for $\overline{d}$

The embedding system is a monotone dynamical system on $\mathbb{R}^{2n}$ with respect to the **southeast** partial order $\leq_{\mathrm{SE}}$:

$$\begin{bmatrix} x \\ \widehat{x} \end{bmatrix} \leq_{\mathrm{SE}} \begin{bmatrix} y \\ \widehat{y} \end{bmatrix} \quad \Longleftrightarrow \quad x \leq y \quad \text{and} \quad \widehat{y} \leq \widehat{x}$$

In terms of cones, $\leq_{\mathrm{SE}}$ is induced by the cone $\mathbb{R}^n_{\geq 0} \times -\mathbb{R}^n_{\geq 0}$.

- f locally Lipschitz $\implies$ a decomposition function exists
- decomposition functions are not unique (use structure of the system to construct one)

History:

J-L. Gouze and L. P. Hadeler. Monotone flows and order intervals. Nonlinear World, 1994

G. Enciso, H. Smith, and E. Sontag. Nonmonotone systems decom- posable into monotone systems with negative feedback . Journal of Differential Equations, 2006.

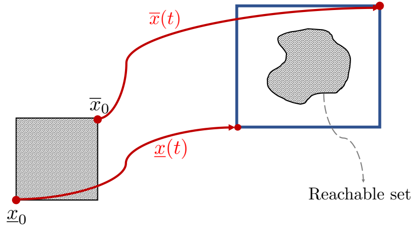
H. Smith. Global stability for mixed monotone systems. Journal of Difference Equations and Applications, 2008

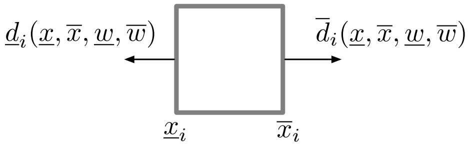
Theorem

Assume $\mathcal{W} = [\underline{w}_0, \overline{w}_0]$ and $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ and

$$\dot{\underline{x}} = \underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}), \qquad \underline{x}(0) = \underline{x}_0$$
$$\dot{\overline{x}} = \overline{d}(\overline{x}, \underline{x}, \overline{w}, \underline{w}), \qquad \overline{x}(0) = \overline{x}_0$$

Then $\mathcal{R}^f(t, \mathcal{X}_0) \subseteq [\underline{x}(t), \overline{x}(t)]$





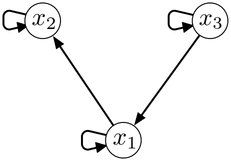
(Scalable) a single trajectory of embedding system provides **lower bound** ($\underline{x}$) and **upper bound** ($\overline{x}$) for the trajectories of the original system.

Embedding System for Linear Dynamical System

A structure preserving decomposition

- **Metzler/non-Metzler** decomposition: $A = \lceil A \rceil^{\mathrm{Mzl}} + \lfloor A \rfloor^{\mathrm{Mzl}}$
- Example: $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \lceil A \rceil^{\mathrm{Mzl}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \lfloor A \rfloor^{\mathrm{Mzl}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Linear systems

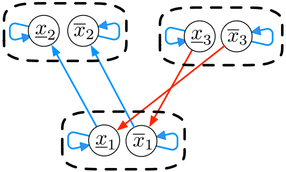


Original system

$$\dot{x} = Ax + Bw$$

Embedding system

$$\dot{\underline{x}} = \lceil A \rceil^{\mathrm{Mzl}} \underline{x} + \lfloor A \rfloor^{\mathrm{Mzl}} \overline{x} + B^{+} \underline{w} + B^{-} \overline{w}$$
$$\dot{\overline{x}} = \lceil A \rceil^{\mathrm{Mzl}} \overline{x} + \lfloor A \rfloor^{\mathrm{Mzl}} \underline{x} + B^{+} \overline{w} + B^{-} \underline{w}$$



Reachability using Embedding Systems

Example

Original System:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} {\color{red}x_2^3} {\color{blue}- x_2} + {\color{red}w} \\ {\color{red}x_1} \end{bmatrix}$$

$$\mathcal{W} = [2.2\ ,\ 2.3] \quad \mathcal{X}_0 = \left[\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right]$$

red $=$ cooperative, blue $=$ competitive

Embedding System:

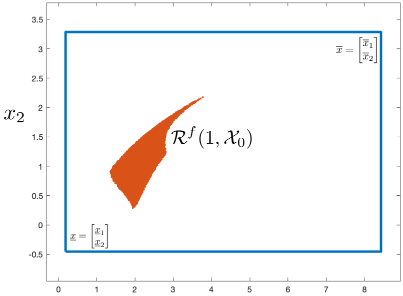
$$\frac{d}{dt}\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = \begin{bmatrix} \underline{x}_2^3 - \overline{x}_2 + \underline{w} \\ \underline{x}_1 \\ \overline{x}_2^3 - \underline{x}_2 + \overline{w} \\ \overline{x}_1 \end{bmatrix} \qquad \begin{bmatrix} \underline{w} \\ \overline{w} \end{bmatrix} = \begin{bmatrix} 2.2 \\ 2.3 \end{bmatrix}$$

$$\begin{bmatrix} \underline{x}_1(0) \\ \underline{x}_2(0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \qquad \begin{bmatrix} \overline{x}_1(0) \\ \overline{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Decomposition function

$$\underline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \begin{bmatrix} \underline{x}_2^3 + \underline{w} \\ \underline{x}_1 \end{bmatrix} + \begin{bmatrix} -\overline{x}_2 \\ 0 \end{bmatrix}$$

$$\overline{d}(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \begin{bmatrix} \overline{x}_2^3 + \overline{w} \\ \overline{x}_1 \end{bmatrix} + \begin{bmatrix} -\underline{x}_2 \\ 0 \end{bmatrix}$$



Decomposition functions

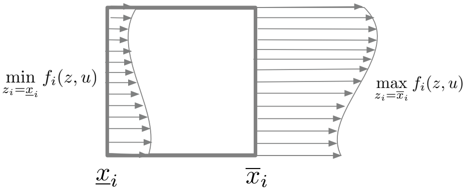
How to decompose a vector field?

Theorem

The best (tightest) decomposition function is given by

$$\underline{d}_i(\underline{x},\overline{x},\underline{w},\overline{w}) = \min_{\substack{z\in[\underline{x},\overline{x}],z_i=\underline{x}_i \\ u\in[\underline{w},\overline{w}]}} f_i(z,u),$$

$$\overline{d}_i(\underline{x},\overline{x},\underline{w},\overline{w}) = \max_{\substack{z\in[\underline{x},\overline{x}],z_i=\overline{x}_i \\ u\in[\underline{w},\overline{w}]}} f_i(z,u)$$



These optimization problems are, in general, not tractable

1. **Compositional approach:**
   - find the tight decomposition functions for a handful of elementary functions
   - write the vector field as composition of some elementary functions
2. **Jacobian-based approach:**
   - Taylor expansion for the vector field
   - formulas for linear systems

Tightness of Mixed Monotone Bounds

Diagonally weighted ℓ_∞-norm contractions

Theorem

Let $\frac{d}{dt}\begin{bmatrix}\underline{x}\\ \overline{x}\end{bmatrix} = \begin{bmatrix}\underline{d}(\underline{x},\overline{x},\underline{w},\overline{w})\\ \overline{d}(\underline{x},\overline{x},\underline{w},\overline{w})\end{bmatrix} := e(\underline{x},\overline{x},\underline{w},\overline{w})$ be the embedding function with the tight decomposition functions for $\dot{x} = f(x,w)$. For any $\eta \in \mathbb{R}^n_{\geq 0}$

$$\mu_{\infty,\eta}\left(\frac{\partial f}{\partial x}(x,w)\right) \leq c \qquad \text{for all } x, w$$

if and only if

$$\mu_{\infty,\eta\otimes I_2}\left(\frac{\partial e}{\partial[\frac{\underline{x}}{\overline{x}}]}(\underline{x},\overline{x},\underline{w},\overline{w})\right) \leq c, \qquad \text{for all } \underline{x},\overline{x},\underline{w},\overline{w}$$

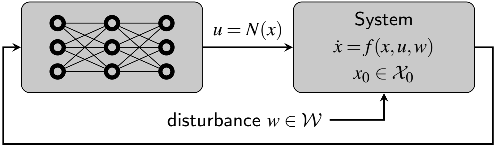
mixed monotone reachability is at least as accurate as contraction reachability wrt diagonally weighted ℓ_∞-norms

Given the open-loop nonlinear system with a neural network controller

$$\dot{x} = f^o(x, u, w),$$
$$u = N(x),$$

study reachability of the closed-loop system

$$\dot{x} = f^o(x, N(x), w) := f^c(x, w)$$



Challenge: finding a tight closed-loop decomposition function

Compositional Approach:

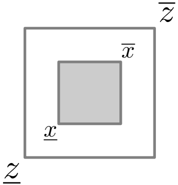
1. compute the open-loop decomposition function $\underline{d}^o, \overline{d}^o$ for $\dot{x} = f^o(x, u, w)$
2. construct the neural network bounding functions $\underline{N}$ and $\overline{N}$
   - bounds from NN verification algorithms (CROWN, LipSDP, IBP, etc)
   - construction is expensive but queries is cheap
3. combine (1) and (2) to obtain a decomposition function for the closed-loop system

Reachability of Neural Network Loops

Neural Network Bounds

Given a neural network controller $u = N(x)$ and an interval $[\underline{z}, \overline{z}]$,

$$\underline{N}_{[\underline{z},\overline{z}]}(\underline{x}, \overline{x}) \leq N(x) \leq \overline{N}_{[\underline{z},\overline{z}]}(\underline{x}, \overline{x}), \qquad \text{for all } [\underline{x}, \overline{x}] \subseteq [\underline{z}, \overline{z}]$$



- CROWN[2] provides affine bounds of the form

$$\underline{A}_{[\underline{z},\overline{z}]} x + \underline{b}_{[\underline{z},\overline{z}]} \leq N(x) \leq \overline{A}_{[\underline{z},\overline{z}]} x + \overline{b}_{[\underline{z},\overline{z}]}, \qquad \text{for all } x \in [\underline{z}, \overline{z}]. \tag{1}$$

Given the CROWN bounds (1), we have

$$\underline{N}_{[\underline{z},\overline{z}]}(\underline{x}, \overline{x}) = \underline{A}^{+}_{[\underline{z},\overline{z}]} \underline{x} + \overline{A}^{-}_{[\underline{z},\overline{z}]} \underline{x} + \underline{b}_{[\underline{z},\overline{z}]},$$

$$\overline{N}_{[\underline{z},\overline{z}]}(\underline{x}, \overline{x}) = \overline{A}^{+}_{[\underline{z},\overline{z}]} \overline{x} + \underline{A}^{-}_{[\underline{z},\overline{z}]} \underline{x} + \overline{b}_{[\underline{z},\overline{z}]}$$

[2] Zhang, Weng, Chen, Hsieh,Daniel, "Efficient neural network robustness certification with general activation functions." NeurIPS, 2018.

Theorem

Given open-loop decomposition functions $\underline{d}^o, \overline{d}^o$ for $\dot{x} = f^o(x, u, w)$ and the neural network bounds $\underline{N}_{[\underline{z},\overline{z}]}, \overline{N}_{[\underline{z},\overline{z}]}$ for the neural network controller $u = N(x)$. Then

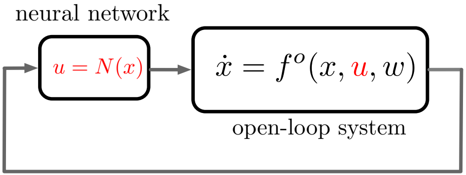
$$\underline{d}_i^c(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \underline{d}_i^o(\underline{x}, \overline{x}, \underline{\eta}^i, \overline{\eta}^i, \underline{w}, \overline{w})$$
$$\overline{d}_i^c(\underline{x}, \overline{x}, \underline{w}, \overline{w}) = \overline{d}_i^o(\underline{x}, \overline{x}, \underline{\nu}^i, \overline{\nu}^i, \underline{w}, \overline{w})$$

where

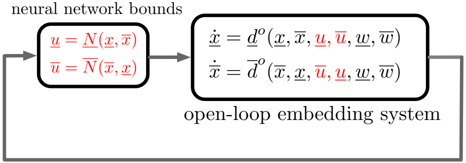
$$\underline{\eta}^i = \underline{N}_{[\underline{x},\overline{x}]}(\underline{x}, \overline{x}_{[i:\underline{x}]}) \quad \overline{\eta}^i = \overline{N}_{[\underline{x},\overline{x}]}(\underline{x}, \overline{x}_{[i:\underline{x}]}),$$
$$\underline{\nu}^i = \underline{N}_{[\underline{x},\overline{x}]}(\underline{x}_{[i:\overline{x}]}, \overline{x}) \quad \overline{\nu}^i = \overline{N}_{[\underline{x},\overline{x}]}(\underline{x}_{[i:\overline{x}]}, \overline{x}),$$

are decomposition functions for the closed-loop system where $v_{[i:w]}$ is the vector v with ith component replaced with ith component of w.

Original system:

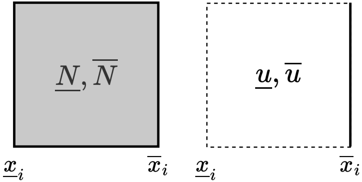


Embedding system:



How does the interconnection work?

- NN bounds $\underline{N}_{[\underline{x},\overline{x}]}, \overline{N}_{[\underline{x},\overline{x}]}$ are constructed on $[\underline{x}, \overline{x}]$
- NN bounds are evaluated on each edge, i.e., $\underline{u} = \underline{N}_{[\underline{x},\overline{x}]}(\underline{x}, \overline{x}_{[i:\underline{x}]})$ and $\overline{u} = \overline{N}_{[\underline{x},\overline{x}]}(\underline{x}, \overline{x}_{[i:\underline{x}]})$



Conclusions

- mixed monotone and contraction-based reachability
- connection between these two reachability approaches
- compositional approach for reachability analysis of NN controlled systems

Example: Bicycle Model

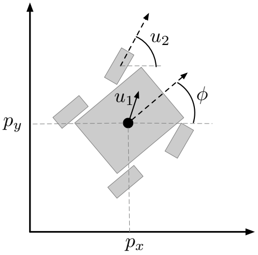
Design of the neural network

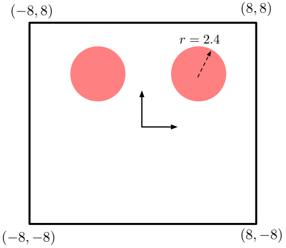
Dynamics of bicycle

$$\dot{p_x} = v\cos(\phi + \beta(u_2)) \qquad \dot{\phi} = \frac{v}{\ell_r}\sin(\beta(u_2))$$

$$\dot{p_y} = v\sin(\phi + \beta(u_2)) \qquad \dot{v} = u_1$$

$$\beta(u_2) = \arctan\left(\frac{l_r}{l_f + l_r}\tan(u_2)\right)$$





Goal: steer the bicycle to the origin avoiding the obstacles

- **offline controller:** MPC with hard constraint to avoid the obstacles
- run MPC for 65000 randomly chosen initial condition (20 sample per trajectory)
- train a feedforward neural network $4 \mapsto 100 \mapsto 100 \mapsto 2$ with this data

- start from $(8,8)$ toward $(0,0)$
- $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ with

$$\underline{x}_0 = \begin{pmatrix} 7.9 & 7.9 & -\frac{2\pi}{3} - 0.01 & 1.99 \end{pmatrix}^\top$$
$$\overline{x}_0 = \begin{pmatrix} 8.1 & 8.1 & -\frac{2\pi}{3} + 0.01 & 2.01 \end{pmatrix}^\top$$

- CROWN for verification of neural network
- partition the states to improve accuracy
- mixed monotone approach **certify** that closed-loop system is avoiding the obstacle