Weak and Semi-contractions for Large-Scale Network Systems

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SJ and P. Cisneros-Velarde and F. Bullo. Weak and Semi-Contraction for Network Systems and Diffusively-Coupled Oscillators. IEEE Transactions on Automatic Control, Mar. 2021.

A. Davydov and SJ and F. Bullo. Non-Euclidean Contraction Theory for Robust Nonlinear Stability. IEEE Transactions on Automatic Control, Dec. 2022

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Large-scale nonlinear networks

Introduction

Transportation networks

Brain neural network

Learning-based systems

- large penetration of inteligent units in power and transportation networks
- increasing deployment of neural networks in safety-critical systems
- Brain neural networks consist of billions of neurons interacting with each other

societal autonomous systems are becoming **large-scale** with **interconnected** and **nonlinear** components

Many networks in nature are extremely large and nonlinear

Goal: to **analyze**, **monitor**, and **control** these large-scale networks

What are the issues with the *classical stability and control* approaches?
(Lyapunov-based methods)

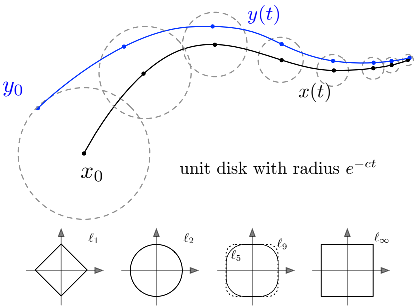
1. computing the equilibria or operating points
    - computationally heavy for large-scale networks with **varying parameters**
2. ℓ_2-norm-based conditions
    - LMI and SOS are not scalable for **large** networks
3. reduction to low-dimensional submanifolds
    - **No systematic approach** for convergence to subspaces or submanifolds

Presentation outline

- **non-Euclidean contraction theory**
  - **definition and basic properties**
  - **differential and integral characterizations**
- weakly-contracting systems
  - definition and examples
  - dichotomy in asymptotic behavior
  - example: distributed primal-dual
- semi-contracting systems
  - definition and examples
  - convergence to invariant subspaces
  - example: diffusively-coupled oscillators

Definition (Contraction)

$\dot{x} = f(t,x)$ is contracting wrt $\|\cdot\|$ if
the distance between every two trajectory is decreasing exponentially with rate c wrt $\|\cdot\|$



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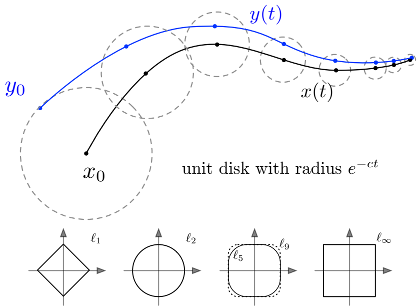
Ordered transient and asymptotic behaviors:

- unique globally exponential stable equilibrium
- efficient equilibrium point computation
- input-output robustness
- modularity and interconnection properties
- $\cdots$



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Differential condition

Logarithmic norm

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a norm $\|\cdot\|$:

$$\mu_{\|\cdot\|}(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

- Directional derivative of norm $\|\cdot\|$ in direction of A,

$$\mu_2(A) = \tfrac{1}{2}\lambda_{\mathsf{max}}(A + A^\top)$$

$$\mu_1(A) = \max_j \left(a_{jj} + \sum\nolimits_{i\neq j} |a_{ij}|\right)$$

$$\mu_\infty(A) = \max_i \left(a_{ii} + \sum\nolimits_{j\neq i} |a_{ij}|\right)$$

[1] A. Davydov, S. Jafarpour, F. Bullo, TAC 2022

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Integral condition

Weak pairing[1]

Given a norm $\|\cdot\|$, the associated weak pairing is $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$:

- Subadditive and weakly homogeneity
- Positive definite
- Cauchy-Schwarz inequality
- $[\![x,x]\!] = \|x\|^2$

$$[\![x,y]\!]_2 = y^\top x$$
$$[\![x,y]\!]_1 = \mathrm{sign}(y)^\top x$$
$$[\![x,y]\!]_\infty = \max\nolimits_{i\in I_\infty(x)} x_i y_i$$

$$I_\infty(x) = \{i \mid |x_i| = \|x\|_\infty\}$$

[1] A. Davydov, S. Jafarpour, F. Bullo, TAC 2022

Theorem[2]

$\dot{x} = f(t,x)$ is contracting wrt $\|\cdot\|$ with rate c iff

Differential: $\mu_{\|\cdot\|}(D_x f(t,x)) \leq -c,$ for all x, t

Integral: $[\![f(t,x) - f(t,y), x - y]\!] \leq -c\|x-y\|^2,$ for all x, y, t

[2] A. Davydov, S. Jafarpour, F. Bullo, TAC 2022

Non-Euclidean contraction theory

Differential and Integral characterizations

Theorem

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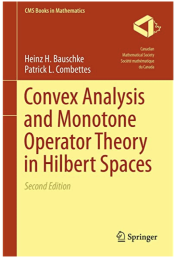
- Connection between **contraction theory** and **monotone operator theory**

f is a contracting vector field wrt to $\|\cdot\|_2$
iff
$-f$ is a strongly monotone operator wrt to the inner product $\langle \cdot, \cdot \rangle$.



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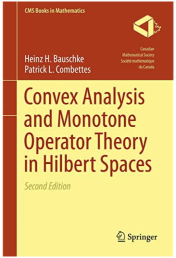

Challenge: many real-world networks are not contracting.



conservation law: $\mathbb{1}_n^\top x(t) = \text{const}$

invariance, symmetry: $f(x + \alpha \mathbb{1}_n) = f(x)$

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For a vector field f and positive vectors $\eta, \xi \in \mathbb{R}^n_{\geq 0}$,

conservation law	$\eta^\top f(x) = \eta^\top f(y) \;\; \forall x, y$	$\Longleftrightarrow$	$\eta^\top D_x f(x) = 0 \;\; \forall x$
translation invariance	$f(x + \alpha\xi) = f(x) \;\; \forall x, \alpha$	$\Longleftrightarrow$	$D_x f(x)\xi = 0 \;\; \forall x$

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translation invariance	$f(x+\alpha\xi) = f(x) \;\; \forall x, \alpha$	$\iff$	$D_x f(x)\xi = 0 \;\; \forall x$

If f satisfies a conservation or resp. invariance, then

1. $\mu(D_x f) \geq 0$,
2. if, additionally, f is cooperative, then $\mu_{1,[\eta]}(D_x f) = 0$ or resp. $\mu_{\infty,[\xi]^{-1}}(D_x f) = 0$

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  - differential and integral characterizations
- **weakly-contracting systems**
  - **definition and examples**
  - **dichotomy in asymptotic behavior**
  - **example: distributed primal-dual**
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Definition: Weakly-contracting systems

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1. Lotka-Volterra population dynamics (Lotka, 1920; Volterra, 1928) (ℓ_1-norm)
2. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981) (ℓ_1-norm and ℓ_∞-norm)
3. Daganzo's cell transmission model for traffic networks (Daganzo, 1994), (ℓ_1-norm)
4. compartmental systems in biology, medicine, and ecology (Sandberg, 1978; Maeda et al., 1978). (ℓ_1-norm)
5. saddle-point dynamics for optimization of weakly-convex functions (Arrow et al., 1958). (ℓ_2-norm)

What is the *asymptotic behavior* of weakly-contracting systems?

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Classical Theorem

$\dot{x} = f(x)$ is contracting, then

- f has a unique globally asymptotically stable equilibrium x^*

What is the *asymptotic behavior* of weakly-contracting systems?

Theorem: Dichotomy

$\dot{x} = f(x)$ is weakly-contracting, then either

1. f has no equilibrium and every trajectory is unbounded, or
2. f has at least one equilibrium x^* and every trajectory is bounded.

Theorem

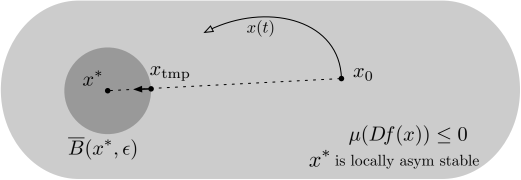
$\dot{x} = f(x)$ is weakly-contracting with at least one equilibrium point x^*:

(i) each equilibrium is stable

(ii) if $\|\cdot\|$ is a polyhedral norm, then every trajectory converges to the set of equilibria,

(iii) x^* is locally asymptotically stable $\implies$ x^* is globally asymptotically stable.

Idea of the proof



Example: Primal-dual algorithm

Distributed implementation over networks

Optimization problem: $\min_{x\in\mathbb{R}^k} f(x) = \min_{x\in\mathbb{R}^k} \sum_{i=1}^{n} f_i(x)$.

Distributed implementation

- n agents locally minimize f and communicate over a undirected weighted graph G,
- agent i have access to function f_i and can exchange x_i with its neighbors.

$$\min_{x\in\mathbb{R}^k} \quad \sum_{i=1}^{n} f_i(x_i)$$

$$x_1 = x_2 = \ldots = x_n$$

In matrix form by assuming $x = (x_1^\top, \ldots, x_n^\top)^\top \in \mathbb{R}^{nk}$:

$$\min_{x\in\mathbb{R}^k} \quad \sum_{i=1}^{n} f_i(x_i)$$

$$(L \otimes I_k)x = \mathbb{0}_{nN}$$

If each f_i is continuously differentiable in x_i:

Lagrangian

$$\mathcal{L}(x,\nu) = \sum_{i=1}^{n} f_i(x_i) + \nu^\top (L \otimes I_k) x$$

Distributed primal-dual algorithm (component form):

$$\dot{x}_i = -\frac{\partial \mathcal{L}}{\partial x_i} = -\nabla f_i(x_i) - \sum_{j=1}^{n} a_{ij}(\nu_i - \nu_j),$$

$$\dot{\nu}_i = \frac{\partial \mathcal{L}}{\partial \nu_i} = \sum_{j=1}^{n} a_{ij}(x_i - x_j)$$

Distributed primal-dual algorithm (vector form):

$$\dot{x} = -\frac{\partial \mathcal{L}}{\partial x} = -\nabla f(x) - (L \otimes I_k)\nu,$$
$$\dot{\nu} = \frac{\partial \mathcal{L}}{\partial \nu} = (L \otimes I_k)x$$

$$Dg(x,\nu) + Dg(x,\nu)^\top = \begin{bmatrix} -\nabla^2 f(x) & -(L \otimes I_k) \\ (L \otimes I_k) & \mathbb{0} \end{bmatrix} + \begin{bmatrix} -\nabla^2 f(x) & (L \otimes I_k) \\ -(L \otimes I_k) & \mathbb{0} \end{bmatrix} = \begin{bmatrix} -\nabla^2 f(x) & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix}$$

$$f \text{ is convex} \implies \mu_2(Dg(x,\nu)) = 0$$

1. f is convex and has a global minimum $x^* \in \mathbb{R}^k$,
2. $\nabla^2 f_i(x) \succeq 0$ for all x, and $\nabla^2 f_i(x^*) \succ 0$, and
3. the undirected weighted graph G is connected with Laplacian L.

Theorem

The distributed primal-dual algorithm

1. is weakly-contracting wrt ℓ_2-norm,
2. $(x(t), \nu(t)) \to (\mathbb{1}_n \otimes x^*, \mathbb{1}_n \otimes \nu^*)$, with $\nu^* = \sum_{i=1}^n \nu_i(0)$,
3. exponential convergence rate is $-\alpha_{\text{ess}}\left(\begin{bmatrix} -\nabla^2 f(x^*) & -L \otimes I_k \\ L \otimes I_k & \mathbb{0} \end{bmatrix}\right)$ where

$$\alpha_{\text{ess}}(A) := \max\{\Re(\lambda) \mid \lambda \in \operatorname{spec}(A) \setminus \{0\}\}.$$

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How to study contraction to *subspaces*?

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Definition (Semi-norm)

$|||\cdot|||$ is a *semi-norm* if

1. $|||cv||| = |c||||v|||$, for every $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$;
2. $|||v + w||| \leq |||v||| + |||w|||$, for every $v, w \in \mathbb{R}^n$.

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- **desired submanifold**: $\mathrm{Ker}\, |||\cdot||| = \{v \in \mathbb{R}^n \mid |||v||| = 0\}$.

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- **desired submanifold**: $\mathrm{Ker}\, |||\cdot||| = \{v \in \mathbb{R}^n \mid |||v||| = 0\}$.
- **Example**: for $k < n$, $R \in \mathbb{R}^{k\times n}$, and norm $\|\cdot\|$, we get $|||x|||_R = \|Rx\|$.

Definition (Logarithmic semi-norm)

The **Logarithmic semi-norm** of $A \in \mathbb{R}^{n\times n}$ wrt $|||\cdot|||$:

$$\mu_{|||\cdot|||}(A) = \lim_{h\to 0^+} \frac{|||I_n + hA||| - 1}{h}.$$

- Directional derivative of $|||\cdot|||$ in direction of A.
- if $\mathrm{Ker}\,|||\cdot|||$ is invariant under A then $\Re(\lambda) \leq \mu_{|||\cdot|||}(A)$, for every $\lambda \in \mathrm{spec}_{\mathrm{Ker}\,|||\cdot|||^{\perp}}(A^\top)$.

Definition (Semi-contraction)

$\dot{x} = f(t,x)$ with f continuously differentiable in x is semi-contracting wrt the semi-norm $|||\cdot|||$ with rate $c > 0$:

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1. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981), (ℓ_1-norm)
2. Chua's diffusively-coupled circuits (Wu and Chua, 1995), (ℓ_2-norm)
3. morphogenesis in developmental biology (Turing, 1952), (ℓ_1-norm)
4. Goodwin model for oscillating auto-regulated gene (Goodwin, 1965). (ℓ_1-norm)

- $\dot{x} = f(t,x)$ is semi-contracting wrt the semi-norm $|||\cdot|||$ with rate $c > 0$, and
- **(Affine invariance)**: $f(t, x^* + \operatorname{Ker}|||\cdot|||) \subseteq \operatorname{Ker}|||\cdot|||$ for every t

Theorem

1. for every trajectory $x(t)$,

$$|||x(t) - x^*||| \leq e^{-ct}|||x(0) - x^*|||, \qquad \text{for every } t \geq 0.$$

2. every trajectory converges to $x^* + \operatorname{Ker}|||\cdot|||$.

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- partial contraction (only for ℓ_2-norms): W. Wang and J.-J. E. Slotine. On partial contraction analysis for coupled nonlinear oscillators.
  Biological Cybernetics, 92(1):38–53, 2005
- horizontal contraction (stronger assumptions): F. Forni and R. Sepulchre. A differential Lyapunov framework for contraction analysis.
  IEEE Trans. Autom. Control, 59(3):614–628, 2014

- n agents with states $x_1, \dots, x_n \in \mathbb{R}^k$ and $x = (x_1, \dots, x_n)^\top$
- identical internal dynamics f;
- interconnected by a weighted undirected connected graph G using diffusive coupling

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

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Applications: Biological networks, Chemical reaction systems, neural networks
A canonical model for weakly coupled oscillators

Example: Diffusively-coupled oscillators
Synchronization

- n agents with states $x_1, \dots, x_n \in \mathbb{R}^k$ and $x = (x_1, \dots, x_n)^\top$
- identical internal dynamics f;
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Applications: Biological networks, Chemical reaction systems, neural networks
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Goal: **asym sync** $\lim_{t\to\infty} \|x_i - x_j\| = 0$ for every i, j

For undirected G with Laplacian L:

The orthogonal projection $\Pi : \mathbb{R}^n \to \mathrm{span}\{\mathbb{1}_n\}^\perp$

$$\Pi = I_n - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top = \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix} \succeq 0$$

- $(\Pi \otimes I_k)x$ measures **dissimilarity** of the states x_i
- $\mu_{2,\Pi}(-L) = -\lambda_2(L)$

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Given a norm $\|\cdot\|$, we define

$$|||x|||_{\Pi\otimes I_k} = \|(\Pi \otimes I_k)x\|$$

We have $\mathrm{Ker}_{|||\cdot|||} = \mathbb{1}_n \otimes \mathbb{R}^n =$ **synchronization**

Two main factors in synchronization:

1. contractivity of the internal dynamics
2. strength of the diffusive coupling

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1. contractivity of the internal dynamics
2. strength of the diffusive coupling

local-global mixed norm: $(2,p)$-tensor norm on $\mathbb{R}^{nk} = \mathbb{R}^n \otimes \mathbb{R}^k$

$$\|u\|_{(2,p)} = \inf\Big\{\Big(\sum_{i=1}^{r}\|v^i\|_2^2\|w^i\|_p^2\Big)^{\frac{1}{2}} \;\Big|\; u = \sum_{i=1}^{r} v^i \otimes w^i\Big\}.$$

- **Global norm**: ℓ_2-norm for the interactions between agents
- **Local norm**: ℓ_p-norm for internal dynamics of each agent

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

G is a connected weighted graph with Laplacian L

Theorem

Suppose that

$$\mu_p(Df(t,x)) \leq \lambda_2(L) - c, \qquad \text{for every } t, x$$

then

1. the dynamics is semi-contracting wrt $\|\cdot\|_{(2,p),(\Pi\otimes I_k)}$;
2. for every trajectory $x(t)$,

$$\|x(t) - \mathbb{1}_n \otimes x_{\mathrm{ave}}(t)\|_{(2,p),(\Pi\otimes I_k)} \leq e^{-ct}\|x(0) - \mathbb{1}_n \otimes x_{\mathrm{ave}}(0)\|_{(2,p),(\Pi\otimes I_k)}.$$

3. the system achieves synchronization: $\lim_{t\to\infty} x(t) = \mathbb{1}_n \otimes x_{\mathrm{ave}}(t)$

where $x_{\mathrm{ave}}(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t)$

$$\mu_p(Df(t,x)) \leq \lambda_2(L) - c, \qquad \text{for every } t, x$$

- trade off between internal dynamics and coupling strength

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- trade off between internal dynamics and coupling strength
- f **time-invariant:** every trajectory converges to an equilibrium point in $\mathbb{1}_n \otimes \mathbb{R}^k$.
- f **periodic:** every trajectory converges to a periodic orbit in $\mathbb{1}_n \otimes \mathbb{R}^k$.

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- trade off between internal dynamics and coupling strength
- f **time-invariant:** every trajectory converges to an equilibrium point in $\mathbb{1}_n \otimes \mathbb{R}^k$.
- f **periodic:** every trajectory converges to a periodic orbit in $\mathbb{1}_n \otimes \mathbb{R}^k$.
- Unstable dynamics f, sufficiently strong coupling $\implies$ $\lambda_2(L)$ large $\implies$ the network synchronizes.

Summary

- reviewed classical contraction theory
- characterization of contraction wrt non-Euclidean norms
- two extensions of classical contraction:
  - weak contraction
  - semi-contraction
- dichotomy in asymptotic behavior of weakly-contracting systems
- convergence to invariant subspaces for semi-contracting systems

Future research

- contraction-based compositional analysis of interconnected systems
  - scalable stability certificates using non-Euclidean contraction.
- computing equilibra of contracting and weakly-contracting systems
  - explicit and implicit integration algorithms
  - accelerated convergence.
- optimization algorithms using contraction theory
  - extension to gradient descent algorithms and time-varying algorithms.
  - connection with discrete-time algorithms for optimization.
- robustness of artificial neural networks using contraction theory
  - use contraction condition for input-output robustness.